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Secția I

MATEMATICĂ
MECANICĂ TEORETICĂ
FIZICĂ

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1981

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Secția I

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Section I

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SUR UNE ALGÈBRE RÉELLE NONASSOCIATIVE

PAR

I. ENESCU

1. Dans [1] on présente l'algèbre réelle A donnée par la table

	e_1	e_2	e_3
e_1	0	e_3	e_2
e_2	e_3	0	e_1
e_3	e_2	e_1	0

L'algèbre A est commutative mais elle n'est pas associative. Cette algèbre n'est pas alternative, ni à pouvoirs associatifs et ni Jordan.

Dans l'algèbre A a lieu la relation

$$(2) \quad x^2 x^2 = 8abcx$$

pour tout élément $x = ae_1 + be_2 + ce_3$ de A .

Dans cette Note, nous allons étudier la structure de l'algèbre A .

2. Tout d'abord, nous remarquons que $x(xx) = xx^2 = (xx)x = x^2x = x^3$ et $(xx^2)x = x(xx^2) = xx^3 = x^3x$. On voit, par calcul direct, que, en général, $xx^3 \neq x^2x^2$.

Le calcul du produit xx^3 conduit à

$$xx^3 = 2[(a^2 + b^2 + c^2)(bce_1 + ace_2 + abe_3) + abc(ae_1 + be_2 + ce_3)],$$

ce qui démontre le

Théorème 1. *Tout élément $x = ae_1 + be_2 + ce_3$ de A , satisfait la relation*

$$(3) \quad xx^3 = (a^2 + b^2 + c^2)x^2 + 2abcx.$$

Nous remarquons que la relation (3) est indépendante de (2).

3. En utilisant les relations (2) et (3), nous allons déterminer les éléments x de A qui satisfont la condition $xx^3 = x^2x^2$, ce qui revient à la détermination des solutions dans A de l'équation

$$(4) \quad (a^2 + b^2 + c^2)x^2 + 2abcx = 8abcx.$$

Compte tenu de (2) et (3), on obtient la condition

$$(a^2 + b^2 + c^2)(bce_1 + ace_2 + abe_3) = 3abc(ae_1 + be_2 + ce_3),$$

qui conduit au système d'équations à inconnues réelles a, b, c :

$$(5) \quad bc(a^2 + b^2 + c^2) = 3a^2bc, \quad ac(a^2 + b^2 + c^2) = 3ab^2c, \quad ab(a^2 + b^2 + c^2) = 3abc^2.$$

Par conséquence, compte tenu de (5), les solutions de l'équation (4) sont de la forme $\{\lambda e_1\}$, $\{\lambda e_2\}$, $\{\lambda e_3\}$ et respectivement

$$(6) \quad \begin{aligned} x &= \lambda(e_1 + e_2 + e_3), & x &= \lambda(-e_1 + e_2 + e_3), \\ x &= \lambda(e_1 - e_2 + e_3), & x &= \lambda(-e_1 - e_2 + e_3), \end{aligned}$$

avec λ réel arbitraire.

On constate immédiatement que l'ensemble des éléments $\{\lambda e_i\}$ forme, pour respectivement chaque valeur fixée de $i = \overline{1, 3}$, l'algèbre nulle d'ordre un.

Étudierons la structure des ensembles de solutions (6) et, dans ce but, posons

$$u_1 = e_1 + e_2 + e_3, \quad u_2 = -e_1 + e_2 + e_3, \quad u_3 = e_1 - e_2 + e_3, \quad u_4 = -e_1 - e_2 + e_3.$$

On constate immédiatement que $(\frac{1}{2}u_1)^2 = \frac{1}{2}u_1$, c'est-à-dire l'élément $E_1 = \frac{1}{2}u_1$ est idempotent.

D'autre part, l'ensemble $\{\lambda u_1\}$ forme, dans l'algèbre A , une sousalgèbre d'ordre un. En effet, tout élément du cet ensemble s'exprime par le produit λE_1 , avec λ réel arbitraire et, alors, le produit de deux éléments quelconques $\alpha E_1, \beta E_1$ de l'ensemble $\{\lambda E_1\}$ satisfait la condition $(\alpha E_1)(\beta E_1) = (\alpha\beta)(E_1 E_1) = (\alpha\beta)E_1^2 = (\alpha\beta)E_1 \in \{\lambda E_1\}$. Les autres conditions pour être sousalgèbre de A sont, évidemment, satisfaites. La sousalgèbre $\{\lambda E_1\}$ est isomorphe avec le corps R des nombres réels.

On aboutit à la même conclusion, en utilisant un raisonnement identique, pour respectivement les ensembles de solutions $\{\lambda u_2\}$, $\{\lambda u_3\}$, $\{\lambda u_4\}$. Ici, les idempotents sont respectivement $E_2 = -\frac{1}{2}u_2$, $E_3 = -\frac{1}{2}u_3$, $E_4 = \frac{1}{2}u_4$.

Évidemment, les quatre sousalgèbres $\{\lambda E_j\}$, $j = \overline{1, 4}$, sont isomorphes et, bien sur, elles sont isomorphes avec R .

Les résultats établis peuvent être formulés par le

Théorème 2. *Les seuls éléments de l'algèbre (1) qui satisfont la relation $xx^3 = x^2x^2$ sont de la forme $\{\lambda E_j\}$, $j = \overline{1, 4}$ et, respectivement $\{\lambda e_i\}$, $i = \overline{1, 3}$, $\forall \lambda \in R$. Ils forment, dans A , des sousalgèbres d'ordre un isomorphes avec R et respectivement avec l'algèbre nulle d'ordre un.*

Remarques. 1. Les éléments λe_i , $i = \overline{1, 3}$, sont les seuls nilpotents d'indice deux de l'algèbre (1), comme on peut constater par la résolution dans A de l'équation $x^2 = 0$.

2. Le Théorème 2, corroboré avec la relation (2), démontre que les éléments E_j , $j = \overline{1, 4}$, sont les seuls idempotents différents de zéro de l'algèbre A .

3. L'algèbre A n'a pas d'unité.

Comme une algèbre d'ordre un ne peut être, à l'isomorphisme près, que l'algèbre nulle ou le corps R des scalaires, il en résulte, compte tenu des remarques précédentes, que le Théorème 2 a le suivant

Corollaire. Toutes les sousalgèbres d'ordre un de l'algèbre A sont formées par les solutions de l'équation $xx^3 = x^2x^2$.

Remarques. 1. Aucune sousalgèbre nulle d'ordre un, $\{\lambda e_i\}$, $i = \overline{1, 3}$, n'est pas idéal dans A et dans aucune sousalgèbre de A . En effet, soit, par exemple, la sousalgèbre $\mathcal{N} = \{\lambda e_1\}$ et $x = ae_1 + be_2 + ce_3$ un élément quelconque de A qui n'est contenu dans $\mathcal{N}$. Alors, le produit $(\lambda e_1)x = \lambda e_1(ae_1 + be_2 + ce_3) = \lambda(be_3 + ce_2)$ n'est pas contenu dans $\mathcal{N}$. La même conclusion est, évidemment, valable pour les sousalgèbres $\{\lambda e_2\}$ et $\{\lambda e_3\}$.

3. Pour tout élément $x = ae_1 + be_2 + ce_3$ de A , ont lieu les relations

$$(7) \quad \begin{aligned} x + 2xE_1 &= 2(a+b+c)E_1, & x + 2xE_2 &= 2(a-b-c)E_2, \\ x + 2xE_3 &= 2(-a+b-c)E_3, & x + 2xE_4 &= 2(-a-b+c)E_4 \end{aligned}$$

comme on peut constater directement par calcul.

4. Nous allons maintenant étudier l'existence dans A , des sousalgèbres d'ordre deux. Il existe de telles sousalgèbres; en effet, en considérant la base formée par les idempotents E_1, E_2, E_3 , on obtient la table

$$(8) \quad \begin{array}{c|cc} & E_1 & E_2 & E_3 \\ \hline E_1 & E_1 & -\frac{1}{2}(E_1+E_2) & -\frac{1}{2}(E_1+E_3) \\ E_2 & -\frac{1}{2}(E_1+E_2) & E_2 & -\frac{1}{2}(E_2+E_3) \\ E_3 & -\frac{1}{2}(E_1+E_3) & -\frac{1}{2}(E_2+E_3) & E_3 \end{array}$$

qui met en évidence le fait que, chacune paire d'idempotents E_1, E_2, E_3 forme une base pour respectivement une sousalgèbre d'ordre deux dans A , à savoir

$$(9) \quad \begin{array}{c|cc} & E_1 & E_2 \\ \hline E_1 & E_1 & -\frac{1}{2}(E_1+E_2) \\ E_2 & -\frac{1}{2}(E_1+E_2) & E_2 \end{array} \quad \begin{array}{c|cc} & E_1 & E_3 \\ \hline E_1 & E_1 & -\frac{1}{2}(E_1+E_3) \\ E_3 & -\frac{1}{2}(E_1+E_3) & E_3 \end{array} \quad \begin{array}{c|cc} & E_2 & E_3 \\ \hline E_2 & E_2 & -\frac{1}{2}(E_2+E_3) \\ E_3 & -\frac{1}{2}(E_2+E_3) & E_3 \end{array}$$

Cettes sousalgèbres sont isomorphes tel comme on voit de (9).

L'idempotent E_1 et respectivement chacun nilpotent e_i , $i = \overline{1, 3}$, forment une base pour une sousalgèbre d'ordre deux de A . En effet, en posant $E_1 = v_1$, $e_i = v_2$, i fixé, nous avons

$$(10) \quad \begin{array}{c|cc} & v_1 & v_2 \\ \hline v_1 & v_1 & v_1 - \frac{1}{2}v_2 \\ v_2 & v_1 - \frac{1}{2}v_2 & 0 \end{array}$$

En changeant la base dans (10) par $u_1 = v_1$, $u_2 = -v_1 + v_2$, on aboutit à la table

$$\begin{array}{c|cc} & u_1 & u_2 \\ \hline u_1 & u_1 & -\frac{1}{2}(u_1+u_2) \\ u_2 & -\frac{1}{2}(u_1+u_2) & u_2 \end{array}$$

et on voit que les sousalgèbres (9) et respectivement (10) sont isomorphes. Nous avons ainsi démontré le

Théorème 3. *L'algèbre A contient, à l'isomorphisme près, au moins un type de sousalgèbre d'ordre deux représenté, par exemple, par la table (10).*

5. Examinons l'existence des idéaux propres de l'algèbre (1) et, supposons que $\mathcal{J}$ est un tel idéal de A . Nous distinguons deux cas, à savoir:

1) Un idempotent, par exemple $E_1 \in \mathcal{J}$ et, soit $x = ae_1 + be_2 + ce_3$ arbitraire dans A . Alors, selon (7), il en résulte que $x = 2(a+b+c)E_1 - 2xE_1$, ce qui attire le fait que $x \in \mathcal{J}$ et, par conséquent, $A = \mathcal{J}$.

2) $E_1 \notin \mathcal{J}$ et soit x arbitraire dans $\mathcal{J}$. Alors, selon (7), il en résulte que $x + 2xE_1 = 2(a+b+c)E_1 \in \mathcal{J}$, qui implique $E_1 \in \mathcal{J}$, donc une contradiction.

Nous avons donc démontré le

Théorème 4. *L'algèbre A est simple.*

6. Soit H une sousalgèbre d'ordre deux de A et P un idéal propre de H . Supposons que H contient un idempotent, par exemple E_1 . Il y a deux possibilités:

1) $E_1 \in P$ et, alors, si $x = ae_1 + be_2 + ce_3$ est arbitraire dans H , il en résulte selon (7), que $2(a+b+c)E_1 - 2xE_1 = x \in P$, donc $H = P$.

2) $E_1 \notin P$. En prenant x arbitraire dans P et compte tenu de (7), on aboutit à $x + 2xE_1 = 2(a+b+c)E_1$ ce qui conduit à la contradiction $E_1 \in P$.

Pour les cas dans lesquels H contient tout autre idempotent, par un raisonnement identique, on obtient le même résultat.

Si H ne contient pas aucun idempotent, alors cette sousalgèbre ne peut pas avoir qu'un seul type de sousalgèbre d'ordre un, à savoir l'algèbre nulle qui, compte tenu d'une remarque antérieure, n'est pas idéal pour aucune sousalgèbre de A .

Enfin, compte tenu du fait qu'une algèbre d'ordre un n'a pas des idéaux propres, nous venons de démontrer le

Théorème 5. *Toute sousalgèbre de A est simple.*

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ASUPRA UNEI ALGEBRE REALE NEASOCIATIVE

(Rezumat)

Se consideră algebra neasociativă A dată prin (1), prezentată în [1] și care este comutativă dar nu este nici alternativă, nici cu puteri asociative și nici Jordan. În algebra (1) are loc relația (2), satisfăcând de orice element $x = ae_1 + be_2 + ce_3$ al algebrei A .

Se constată că $xx^3 \neq x^2x^2$ și se arată că are loc relația (3), valabilă pentru orice element x din algebra A .

Se determină elementele algebrei (1) care satisfac ecuația $xx^3 = x^2x^2$, constatându-se că acestea constituie respectiv mulțimile $\{\lambda e_i\}$, $i = 1, 3$, $\forall \lambda \in R$ și (6), mulțimi care formează subalgebrele de ordinul unu ale algebrei A . După ce se arată că elementele E_j , $j = 1, 4$, sînt singurii idempotenți din A , se constată că au loc relațiile (7) valabile pentru orice element x din A .

Se dovedește că în A există cel puțin un tip de subalgebră de ordin doi, reprezentat prin (10) și se arată că algebra (1) nu are nici un ideal propriu (este o algebră simplă); în sfîrșit, se demonstrează că orice subalgebră a lui A este, de asemenea, simplă.

CATÉGORIES DE TRIOS ET DES QUATUORS

PAR

DEKO V. DEKOV

Montrons la non-valabilité du Théorème 2 de [2].

Soit C une catégorie. Soient L et K deux sous-catégories de C telles que $L_0 = K_0$; L_0 (respectivement K_0) désigne la classe d'unités de L (respectivement de K). Soit $\square(K, L)$ la catégorie de trios de (K, L) (voir [1], chap. I, n° 3, C). Soient $K \sqsubseteq L$ et $K \sqsupseteq L$ respectivement la classe des quadruplets et la catégorie latérale des quadruplets de (K, L) ([1], chap. I, n° 3, C).

Théorème ([2], Théorème 2). Il existe une application de $\square(K, L)$ dans $K \sqsupseteq L$.

$$(1) \quad \bar{\tau}': (k', k, h) \rightarrow (h', k', k, h)$$

qui définit un foncteur $\bar{\tau}' = (K \sqsupseteq L, \bar{\tau}', \square(K, L))$ et la catégorie des trios est incluse par $\bar{\tau}'$ dans $K \sqsupseteq L$.

Contre-exemple. Soit C la catégorie ayant deux unités distinctes e et e' et une seule flèche f de source e et de but e' . On obtient

$$\square(C, C) = \{(e, e, e), (e, f, e), (e', e', e'), (e', e, f), (e', f, f), (f, e, e), (f, f, e)\};$$

$$C = \square C \{ (e, e, e, e), (e', e', e', e'), (e', e', f, f), (e', f, f, e), (f, e', e, f), (f, f, e, e) \}.$$

On en déduit qu'il n'existe pas une application de $\square(C, C)$ vers $C \sqsubseteq C$ définie par (1).

Ceci montre la non-valabilité du Théorème 2 de [2].

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Stara Zagora, Bulgarie

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CATEGORII DE TRIPLEȚI ȘI CUADRUPLEȚI

(Rezumat)

Se dă un contra-exemplu care infirmă validitatea teoremei 2 din [2].

C. D. 2122

CATÉGORIES DE TRIOS ET DES QUATUORS

PAR
 DEKO V. DEKOV

Montrons la non-valabilité du Théorème 2 de [2].
 Soit C une catégorie point A et W deux sous-catégories de C tel-
 les que $A \in W$, A (respectivement W) désigne la classe d'unités de A ,
 (respectivement de W). Soit $\square(A, W)$ la catégorie des trios de (A, W)
 (voir [1], chap. I n° 3, C). Soient $A \square W$ et $A \square W$ respectivement la classe
 des quadruplets et la catégorie latérale des quadruplets de (A, W) ([1],
 chap. I n° 3, C).
 Théorème 2 ([2], Théorème 2). Il existe une application de
 $\square(A, W)$ dans $A \square W$

$$(1) \quad \gamma: (A, A, A) \rightarrow (A, A, A)$$

qui définit un foncteur $\gamma: \square(A, W) \rightarrow A \square W$ et la catégorie des trios
 est incluse par γ dans $A \square W$.

Contre-exemple. Soit C la catégorie ayant deux unités dis-
 tinctes e et e' et une seule flèche de source e et de but e' . On obtient

$$\square(C, C) = \{(e, e, e), (e', e', e'), (e, e', e), (e', e, e'), (e, e, e'), (e', e', e')\}$$

$$C \square C = \{(e, e, e), (e', e', e'), (e, e', e), (e', e, e'), (e, e, e'), (e', e', e')\}$$

On en déduit qu'il n'existe pas une application de $\square(C, C)$ vers $C \square C$
 définie par (1).

Ceci montre la non-valabilité du Théorème 2 de [2].

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CATEGORII DE TRIUMI I CHETVORICI

(Résumé)

Se démontre la non-valabilité du théorème 2 du [2].

ASUPRA UNOR ALGEBRE NEASOCIATIVE DE ORDIN TREI (I)

DE

MIRCEA ȘTEFANOVICI

1. Ne propunem studiul algebrelor reale, comutative, de ordin trei, în care există o bază formată numai din nilpotenți de indice doi, oricare două elemente ale bazei generînd o subalgebră proprie. Aceste algebre se pot deci reprezenta printr-o tabelă de forma

$$(1) \quad \begin{array}{c|ccc} & u_1 & u_2 & u_3 \\ \hline u_1 & 0 & \alpha u_1 + \gamma u_2 & \beta u_1 + \varepsilon u_3 \\ u_2 & \alpha u_1 + \gamma u_2 & 0 & \delta u_2 + \zeta u_3 \\ u_3 & \beta u_1 + \varepsilon u_3 & \delta u_2 + \zeta u_3 & 0 \end{array}$$

Algebrele (1) apar de exemplu în [1] asociate unor sisteme de ecuații diferențiale pătratică ce descriu procese de interacțiune biologică.

Se constată prin calcul direct că *algebrele (1), cu excepția algebrei nule de ordinul trei, nu sînt nici asociative, nici Jordan, nici alternative, nici cu puteri asociative.*

În cele ce urmează ne propunem să determinăm tipurile neizomorfe de algebre (1), în cazurile cînd cel mult oricare trei din constantele de structură sînt nenule. Evident, algebra pentru care $\alpha = \beta = \gamma = \delta = \varepsilon = \zeta = 0$ este algebra nulă de ordinul trei.

2. Algebrele care au numai una din constantele de structură diferită de zero, sînt grupate în șase familii, după respectiv fiecare din parametrii $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$. Algebrele din aceeași familie sînt izomorfe între ele. Pentru familia în care numai $\alpha \neq 0$, dată deci prin tabela (2), prin schimbarea de bază

$$(2) \quad \begin{array}{c|ccc} & u_1 & u_2 & u_3 \\ \hline u_1 & 0 & \alpha u_1 & 0 \\ u_2 & \alpha u_1 & 0 & 0 \\ u_3 & 0 & 0 & 0 \end{array}, \quad (3) \quad \begin{array}{c|ccc} & e_1 & e_2 & e_3 \\ \hline e_1 & 0 & e_1 & 0 \\ e_2 & e_1 & 0 & 0 \\ e_3 & 0 & 0 & 0 \end{array}$$

$e_1 = \alpha u_1$, $e_2 = \frac{1}{\alpha} u_2$, $e_3 = u_3$ se ajunge la (3). În mod analog, pentru respectiv celelalte familii se obțin tabele identice cu (3), ceea ce stabilește

Teorema 1. *Algebrele din clasa (1) pentru care numai unul, oricare, din parametrii $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$, este diferit de zero, sînt izomorfe între ele și se pot reprezenta prin (3).*

Observăm că algebra (3) nu conține idempotenți, nici nilpotenți de indice trei, iar elementele e_1 și e_3 respectiv e_2 și e_3 constituie baze pentru algebra nulă de ordin doi.

3. Algebrele care au doi din parametrii $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ diferiți de zero sînt grupate în C_6^2 familii.

Pentru familia în care numai α și $\beta \neq 0$, reprezentată prin (4), prin schimbarea de bază $e_1 = \alpha u_1$, $e_2 = \frac{1}{\alpha} u_2$, $e_3 = \frac{1}{\alpha} u_2 - \frac{1}{\beta} u_3$, obținem tabela (3). Același rezultat se găsește și pentru familiile în care numai γ și $\delta \neq 0$ și respectiv ε și $\zeta \neq 0$.

$$(4) \quad \begin{array}{c|ccc} & u_1 & u_2 & u_3 \\ \hline u_1 & 0 & \alpha u_1 & \beta u_1 \\ u_2 & \alpha u_1 & 0 & 0 \\ u_3 & \beta u_1 & 0 & 0 \end{array}$$

Pentru familia în care α și $\gamma \neq 0$ dată prin tabela (5), prin schimbarea de bază $e_1 = \frac{1}{\gamma} u_1$, $e_2 = \frac{1}{\alpha} u_2$, $e_3 = u_3$, se obține (6).

$$(5) \quad \begin{array}{c|ccc} & u_1 & u_2 & u_3 \\ \hline u_1 & 0 & \alpha u_1 + \gamma u_2 & 0 \\ u_2 & \alpha u_1 + \gamma u_2 & 0 & 0 \\ u_3 & 0 & 0 & 0 \end{array}, \quad (6) \quad \begin{array}{c|ccc} & e_1 & e_2 & e_3 \\ \hline e_1 & 0 & e_1 + e_2 & 0 \\ e_2 & e_1 + e_2 & 0 & 0 \\ e_3 & 0 & 0 & 0 \end{array}$$

Pentru familiile în care β și $\varepsilon \neq 0$ și respectiv δ și $\zeta \neq 0$, prin schimbări de bază analoge se obțin tabelele reductibile la (6) prin permutarea elementelor bazei.

Tipul (6) de algebra are un idempotent $e = \frac{1}{2}(e_1 + e_2)$ și un nilpotent de indice trei $u = e_1 - e_2$, ortogonal cu e . De asemenea, e_1 și e_3 , respectiv e_2 și e_3 , constituie baze pentru algebra nulă de ordin doi. Aceste observații arată că algebra (6) nu este izomorfă cu algebra (3).

Pentru familia în care numai β și $\gamma \neq 0$ dată de (8), prin schimbarea de bază

$$(7) \quad e_1 = \frac{1}{\gamma} u_1, \quad e_2 = \gamma u_2, \quad e_3 = \frac{1}{\beta} u_3$$

se obține tabela (9),

$$(8) \quad \begin{array}{c|ccc} & u_1 & u_2 & u_3 \\ \hline u_1 & 0 & \gamma u_2 & \beta u_1 \\ u_2 & \gamma u_2 & 0 & 0 \\ u_3 & \beta u_1 & 0 & 0 \end{array}, \quad (9) \quad \begin{array}{c|ccc} & e_1 & e_2 & e_3 \\ \hline e_1 & 0 & e_2 & e_1 \\ e_2 & e_2 & 0 & 0 \\ e_3 & e_1 & 0 & 0 \end{array}$$

Pentru familiile în care numai β și $\zeta \neq 0$ și respectiv γ și $\delta \neq 0$, α și $\varepsilon \neq 0$, δ și $\varepsilon \neq 0$, prin schimbări de bază analoge cu (7) se obțin tabele care se reduc la (9) prin renumerotarea elementelor bazei.

Tipul (9) de algebră nu conține idempotenți și nici nilpotenți de indice trei. De asemenea conține algebra nulă de ordinul doi, generată de e_2 și e_3 . Aceste observații arată că algebra dată prin tabela (9) nu este izomorfă cu respectiv algebrele date de (3) și (6).

Pentru familia în care β și $\delta \neq 0$ dată prin (11), prin schimbarea de bază

$$(10) \quad e_1 = \beta u_1, \quad e_2 = u_2, \quad e_3 = \frac{1}{\beta} u_3$$

se obține tabela (12), unde $m = \delta/\beta \neq 0$.

$$(11) \quad \begin{array}{c|ccc} & u_1 & u_2 & u_3 \\ \hline u_1 & 0 & 0 & \beta u_1 \\ u_2 & 0 & 0 & \delta u_2 \\ u_3 & \beta u_1 & \delta u_2 & 0 \end{array} \quad (12) \quad \begin{array}{c|ccc} & e_1 & e_2 & e_3 \\ \hline e_1 & 0 & 0 & e_1 \\ e_2 & 0 & 0 & m e_2 \\ e_3 & e_1 & m e_2 & 0 \end{array}$$

Pentru familiile în care α și $\zeta \neq 0$ și respectiv γ și $\varepsilon \neq 0$, prin schimbări de bază analoage cu (10) se obțin tabele care se reduc la (12) prin permutarea elementelor bazei.

Algebrele date prin (12) nu conțin idempotenți și nici nilpotenți de indice trei. Ele conțin algebra nulă de ordin doi, generată de e_1 și e_2 . Deci aceste algebre nu sînt izomorfe, pentru nici o valoare a parametrului $m \neq 0$ cu respectiv algebrele date de (3) și (6).

Fie φ un operator care reprezintă izomorf algebra (12) pe algebra (9) și să punem

$$(13) \quad \begin{aligned} \varphi(e_1) &= A e_1 + B e_2 + C e_3, \\ \varphi(e_2) &= D e_1 + E e_2 + F e_3, \\ \varphi(e_3) &= G e_1 + H e_2 + K e_3. \end{aligned}$$

Condițiile $\varphi(e_1) \cdot \varphi(e_2) = \varphi(e_2)$ și $\varphi(e_1) \cdot \varphi(e_3) = \varphi(e_1)$ conduc la $D = E = F = 0$, ceea ce este imposibil. Deci, algebrele (12) nu sînt izomorfe, pentru nici o valoare a lui $m \neq 0$, cu algebra (9).

Propoziția 1. *Există o infinitate de tipuri neizomorfe de algebre (12), corespunzătoare valorilor $m \neq 0$. Ele se grupează în perechi după, respectiv, m și $\frac{1}{m}$ ($m \neq 0, 1, -1$), fiecare asemenea pereche reprezentînd cîte un tip neizomorf de algebră (12), tipurile $m=1$ și $m=-1$ fiind unice.*

Demonstrație. Fie m_1 și m_2 două valori ale parametrului $m \neq 0$ și φ un operator care reprezintă izomorf algebra corespunzătoare valorii m_1 pe aceea corespunzătoare valorii m_2 , cu $\varphi(e_i)$ dați de (13), ($i=1, 2, 3$). Condițiile $\varphi(e_1) \cdot \varphi(e_3) = \varphi(e_1)$ și $\varphi(e_2) \cdot \varphi(e_3) = m_2 \varphi(e_2)$ conduc la $C = F = 0$ și la $AK = A, m_1 BK = B, DK = m_2 D, m_1 EK = m_2 E$. Condițiile, $\varphi^2(e_1) = \varphi^2(e_2) = \varphi^2(e_3) = \varphi(e_1)$, $\varphi(e_2) = 0$ conduc în plus la ecuațiile $GK = 0$ și $HK = 0$. Dacă $K = 0$ urmează $A = B = 0$, ceea ce este imposibil deoarece $C = 0$. Deci $K \neq 0$ și atunci $G = H = 0$. Din ecuația $AK = A$ rezultă $K = 1$ sau $A = 0$. Dacă $K = 1$, atunci, din celelalte ecuații ar rezulta sau $m_1 = m_2 = 1$, sau $m_1 = m_2$ și $B = D = 0$, ceea ce conduce la izomorfismul identic, sau $m_1 = 1$ și $B = E = 0$ ceea ce este imposibil deoarece $H = 0$. Dacă $A = 0$, cum B nu poate fi nul (deoarece și

$C=0$), rezultă $K=1/m_1$. Din celelalte ecuații rezultă atunci: sau $D=E=0$, ceea ce este imposibil ($F=0$), sau $m_2=1$ și $D=0$, ceea ce este iarăși imposibil ($A=G=0$), $m_1=m_2=1$ (izomorfismul identic), sau în fine, $E=0$ și $m_1 m_2=1$ (B și D oarecare nenuli), c.c.t.d.

De aici rezultă în concluzia paragrafului 3,

Teorema 2. *Algebrele din clasa (1) care au numai doi din parametrii $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ diferiți de zero sînt grupate în următoarele tipuri neizomorfe: (3), (6), (9) și o infinitate de tipuri (12).*

4. Algebrele care au numai trei din parametrii $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ diferiți de zero sînt grupate în $C_8^3=20$ familii.

Pentru familia în care α, β și $\gamma \neq 0$ dată de (15), prin schimbarea de bază

$$(14) \quad e_1 = \frac{1}{\gamma} u_1, \quad e_2 = \frac{1}{\alpha} u_2, \quad e_3 = \frac{1}{\beta} u_3$$

obținem tabela (16),

$$(15) \quad \begin{array}{c|ccc} & u_1 & u_2 & u_3 \\ \hline u_1 & 0 & \alpha u_1 + \gamma u_2 & \beta u_1 \\ u_2 & \alpha u_1 + \gamma u_2 & 0 & 0 \\ u_3 & \beta u_1 & 0 & 0 \end{array}, \quad (16) \quad \begin{array}{c|ccc} & e_1 & e_2 & e_3 \\ \hline e_1 & 0 & e_1 + e_2 & e_1 \\ e_2 & e_1 + e_2 & 0 & 0 \\ e_3 & e_1 & 0 & 0 \end{array}$$

Pentru familiile în care numai γ, δ și $\zeta \neq 0$, α, β și $\varepsilon \neq 0$, β, ε și $\zeta \neq 0$, α, γ și $\delta \neq 0$, δ, ε și $\zeta \neq 0$, prin schimbări de bază analoge cu (14) se obțin tabele care se reduc la (16) prin renumerotarea elementelor bazei.

Algebra (16) are un idempotent $e = \frac{1}{2}(e_1 + e_2)$ și un nilpotent de indice trei $u = e_1 + e_2$ ortogonal cu e . Ea conține algebra nulă de ordinul doi (generată de e_2 și e_3). Deci algebra (16) nu este izomorfă cu algebrele anterior găsite (3), (6), (9) și (12).

Pentru familia în care numai α, δ și $\varepsilon \neq 0$ dată prin tabela (17), prin schimbarea de bază $e_1 = \frac{1}{\varepsilon} u_1$, $e_2 = \frac{1}{\alpha} u_2$, $e_3 = \frac{1}{2} u_3$ se obține tabela (18),

$$(17) \quad \begin{array}{c|ccc} & u_1 & u_2 & u_3 \\ \hline u_1 & 0 & \alpha u_1 & \varepsilon u_3 \\ u_2 & \alpha u_1 & 0 & \delta u_2 \\ u_3 & \varepsilon u_3 & \delta u_2 & 0 \end{array}, \quad (18) \quad \begin{array}{c|ccc} & e_1 & e_2 & e_3 \\ \hline e_1 & 0 & e_1 & e_3 \\ e_2 & e_1 & 0 & e_2 \\ e_3 & e_3 & e_2 & 0 \end{array}$$

Pentru familia în care numai β, γ și $\zeta \neq 0$ prin schimbarea de bază $e_1 = \frac{1}{\zeta} u_3$, $e_2 = \frac{1}{\beta} u_2$, $e_3 = \frac{1}{\gamma} u_1$, obținem aceeași tabelă (18).

Algebra (18) are un idempotent $e = \frac{1}{2}(e_1 + e_2 + e_3)$, nu are nilpotenți de indice trei și nu conține algebra nulă de ordin doi. Deci algebra (18) nu este izomorfă cu nici unul din tipurile găsite anterior (3), (6), (9), (12) și (16).

Pentru familia în care numai α, β și $\delta \neq 0$ dată de (20), prin schimbarea de bază

$$(19) \quad e_1 = u_1, \quad e_2 = \frac{1}{\alpha} u_2, \quad e_3 = \frac{1}{\beta} u_3,$$

se obține tabela (21), unde $m = \beta \neq 0$,

$$(20) \quad \begin{array}{c|ccc} & u_1 & u_2 & u_3 \\ \hline u_1 & 0 & \alpha u_1 & \beta u_1 \\ u_2 & \alpha u_1 & 0 & \delta u_2 \\ u_3 & \beta u_1 & \delta u_2 & 0 \end{array}$$

$$(21) \quad \begin{array}{c|ccc} & e_1 & e_2 & e_3 \\ \hline e_1 & 0 & e_1 & e_1 \\ e_2 & e_1 & 0 & me_2 \\ e_3 & e_1 & me_2 & 0 \end{array}$$

Pentru familiile în care numai α , ϵ și $\zeta \neq 0$ și respectiv γ , δ și $\epsilon \neq 0$, α , β și $\zeta \neq 0$, β , γ și $\delta \neq 0$, γ , ϵ și $\zeta \neq 0$, prin schimbări de bază analoage cu (19), se obțin tabele care se reduc la (21) prin renumerotarea elementelor bazei.

Algebra dată de (21) nu conține idempotenți, nilpotenți de indice trei și nici algebra nulă de ordinul doi, pentru $m \neq 0$. Deci ea nu este izomorfă cu nici unul din tipurile găsite anterior (3), (6), (9), (12), (16) și (18), oricare ar fi $m \neq 0$.

Fie m_1 și m_2 două valori ale parametrului $m \neq 0$ și φ un operator care reprezintă izomorf algebra (21) corespunzătoare valorii m_1 pe aceea corespunzătoare valorii m_2 , cu $\varphi(e_i)$ dați de (13), ($i=1, 2, 3$). Din condițiile $\varphi(e_1) \cdot \varphi(e_2) = \varphi(e_1) \cdot \varphi(e_3) = \varphi(e_1)$ și $\varphi(e_2) \cdot \varphi(e_3) = m_2 \varphi(e_2)$ obținem $C=F=0$, apoi $B=0$, precum și ecuațiile $A(E-1) = A(H+K-1) = D(H+K-m_2) = E(m_1K-m_2) = 0$. Condițiile $\varphi^2(e_1) = \varphi^2(e_2) = \varphi^2(e_3) = 0$ ne dau în plus $DE = -G(H+K) = HK = 0$. Cum K nu poate fi nul, deoarece $C=F=0$, rezultă $H=0$ și $G=0$. Atunci nici E nu poate fi nul, căci $B=H=0$ și deci $D=0$. Revenind la primul grup de ecuații, cum nici A nu poate fi zero ($B=C=0$), rezultă $E=K=1$ și $m_1=m_2$ (A oarecare nenul) ceea ce conduce, în acest caz, la izomorfismul identic. Am demonstrat, deci

Propoziția 2. Există o infinitate de tipuri neizomorfe de algebre (21) corespunzătoare valorilor distincte ale parametrului $m \neq 0$.

Pentru familia în care α , γ și $\epsilon \neq 0$ dată prin tabela (23), prin schimbarea de bază

$$(22) \quad e_1 = \frac{1}{\gamma} u_1, \quad e_2 = \frac{1}{\alpha} u_2, \quad e_3 = u_3$$

se obține tabela (24), unde $m = \epsilon/\gamma \neq 0$,

$$(23) \quad \begin{array}{c|ccc} & u_1 & u_2 & u_3 \\ \hline u_1 & 0 & \alpha u_1 + \gamma u_2 & \epsilon u_3 \\ u_2 & \alpha u_1 + \gamma u_2 & 0 & 0 \\ u_3 & \epsilon u_3 & 0 & 0 \end{array}$$

$$(24) \quad \begin{array}{c|ccc} & e_1 & e_2 & e_3 \\ \hline e_1 & 0 & e_1 + e_2 & me_3 \\ e_2 & e_1 + e_2 & 0 & 0 \\ e_3 & me_3 & 0 & 0 \end{array}$$

Pentru familiile în care numai α , δ și $\zeta \neq 0$ și respectiv β , δ și $\epsilon \neq 0$, α , γ și $\zeta \neq 0$, β , γ și $\epsilon \neq 0$, β , δ și $\zeta \neq 0$, prin schimbări de bază analoage cu (22) se obțin tabele reducibile la (24) prin renumerotarea elementelor bazei.

Algebrele date de (24) conțin un idempotent $e = \frac{1}{2}(e_1 + e_2)$ și un nilpotent de indice trei $u = e_1 - e_2$ ortogonal cu e . De asemenea conțin algebra nulă de ordinul doi (generată de e_2 și e_3). Deci aceste algebre nu sînt izomorfe cu algebrele date prin (3), (6), (9), (12), (18) și (21), pentru orice $m \neq 0$. Presupunem că φ este un operator care reprezintă izomorf algebra (24) pe algebra (16) și punem $\varphi(e_i)$ ca în formulele (13), ($i=1, 2, 3$). Din condițiile $\varphi^2(e_1) = \varphi^2(e_2) = \varphi^2(e_3) = 0$ rezultă sistemul

$$(25) \quad AB=AC=DE=DF=GH=GK=0.$$

Deoarece operatorul φ este nedegenerat, sistemul (25) are doar soluțiile:

$$(a) \quad A=E=F=G=0; \quad (b) \quad A=D=H=K=0; \quad (c) \quad B=C=D=G=0.$$

Dar din condiția $\varphi(e_1) \cdot \varphi(e_3) = \varphi(e_1)$ rezultă ecuațiile $AH+BG=A=B$, $(AK+CG)m=C$. Atunci soluția (a) nu este posibilă, deoarece rezultă $B=C=0$, de asemenea (c) nu este posibilă, rezultând $A=0$. Soluția (b) rămâne în discuție, ultimele ecuații fiind satisfăcute pentru $B=0$ și $CGm=G$. Însă din condiția $\varphi(e_1) \cdot \varphi(e_2) = \varphi(e_1) + \varphi(e_2)$ rezultă ecuațiile $AE+BD=A+D=B+E$, $(AF+CD)m=C+F$, de unde și $E=0$, ceea ce este imposibil ($A=B=D=E=0$).

Așadar algebrele (24) nu sînt izomorfe pentru nici o valoare a lui $m \neq 0$ cu algebra (16).

Propoziția 3. *Există o infinitate de tipuri neizomorfe de algebre (24) corespunzătoare valorilor distincte ale parametrului nenul m .*

Demonstrație. Fie m_1 și m_2 două valori ale parametrului $m \neq 0$ și φ un operator care reprezintă izomorf algebra (24) corespunzătoare valorii m_1 pe aceea corespunzătoare valorii m_2 . Să luăm $\varphi(e_i)$ la fel ca în (13), ($i=1, 2, 3$). Din condițiile $\varphi^2(e_1) = \varphi^2(e_2) = \varphi^2(e_3) = 0$ rezultă același sistem (25) cu soluțiile (a), (b), (c). Dar din condiția $\varphi(e_1) \cdot \varphi(e_3) = m_2 \varphi(e_3)$ rezultă ecuațiile $AH+BG=m_2G=m_2H$, $m_1(AK+CG)=m_2K$. Atunci soluția (a) nu este posibilă deoarece rezultă $H=0$ și $K=0$ (avem și $E=F=0$), de asemenea (b) nu este posibilă rezultînd $G=0$ ($H=K=0$). Rămîne de analizat soluția (c) la care se adaugă $H=0$ și $m_1=m_2$ (deoarece K nu poate fi nul). Aceasta ne conduce la izomorfismul identic, și propoziția este demonstrată.

Astfel, este stabilit următorul rezultat:

Teorema 3. *Algebrele din clasa (1) care au numai trei din parametrii $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ diferiți de zero sînt grupate în următoarele tipuri neizomorfe: (16), (18), o infinitate de tipuri (21) și o infinitate de tipuri (24).*

Cazurile cînd patru, cinci sau șase constante de structură sînt nenule vor constitui obiectul unei viitoare note.

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О НЕКОТОРЫХ НЕАССОЦИАТИВНЫХ ТРЕХМЕРНЫХ АЛГЕБРАХ (1)

(Резюме)

В данной работе определяются неизоморфные типы алгебр (1), в случаях когда не более трёх (любых) параметров отличны от нуля. Выделяются алгебры (3), (6), (9), (12), (16), (18), (21), (24) и выявляются некоторые структурные свойства этих алгебр.

L'ÉTUDE DE LA CLASSE $\Phi_+(X)$ À L'AIDE DE LA MESURE DE NON-COMPACTITÉ

PAR

OLGA COSTINESCU et CĂTĂLINA DAVIDEANU

1. Le but de cette Note est de caractériser la classe des opérateurs semi-Fredholm, à l'aide de la mesure de non-compacité.

Nous précisons, d'abord, les notions et les notations qui seront utilisées pour établir le résultat central (le Théorème 1).

Dans ce qui suit, X indique un espace de Banach réel, $B(X)$ est l'algèbre des opérateurs linéaires et bornés, $T: X \rightarrow X$, $K(X)$, $B(X)$ est l'idéal des opérateurs compacts, S_x indique la sphère unité dans l'espace X , $M(X)$ est la classe des ensembles bornés de X et $\mathcal{X} \subset \mathcal{P}(X)$ la famille des ensembles compacts dans l'espace X .

2. Définition 1. Soit $E \in M(X)$: on appelle mesure de non-compacité de E l'élément $q(E)$, donné par $\inf \{r \mid r \in \mathbb{R}_+, \text{ tel qu'il existe un recouvrement fini pour } E, \text{ avec des sphères de rayon } r\}$ (voir [3]).

On démontre, sans difficulté, la

Proposition. 1. $q(E) = 0$ si et seulement si $\bar{E} \in \mathcal{X}$.

2. Quel que soient $E_1, E_2 \in M(X)$, où $E_1 \subset E_2$ on a $q(E_1) \leq q(E_2)$.

3. Quel que soient $E_1, E_2 \in M(X)$ on a la relation $q(E_1 + E_2) \leq q(E_1) + q(E_2)$.

4. Quel que soient $\alpha \in \mathbb{R}$ et $E \in M(X)$ on a $q(\alpha E) = |\alpha| q(E)$.

5. $\dim X < \infty$ si et seulement si $q(S_x) = 0$ (v. [5]).

Il faut remarquer que si X est de dimension infinie, alors $q(S_x) = 1$ (v. [4]).

Définition 2. Soit $T \in B(X)$; on appelle mesure de non-compacité pour l'opérateur T l'élément $q[T(S_x)]$; on indique cet élément par $\|T\|_q$.

Il est facile à démontrer la suivante

Proposition 2.1. L'application $\|\cdot\|_q: B(X) \rightarrow \mathbb{R}$ est une seminorme.

2. $T \in K(X)$ si et seulement si $\|T\|_q = 0$.

3. $\|T + K\|_q = \|T\|_q$ quel que soient $T \in B(X)$ et $K \in K(X)$.

4. $\|T_1 T_2\|_q \leq \|T_1\|_q \|T_2\|_q$ quel que soient $T_1, T_2 \in B(X)$.

Définition 3. L'opérateur $T \in B(X)$ s'appelle semi-Fredholm s'il satisfait aux conditions suivantes;

- a) $R(T) = \overline{R(T)}$,
 b) $\dim N(T) < \infty$, ou
 b') $\dim X /_{R(T)} < \infty$.

Soient

$\Phi_+(X) = \{T \mid T \in B(X), T \text{ satisfait aux conditions (a) et (b)}\}$

$\Phi_-(X) = \{T \mid T \in B(X), T \text{ satisfait aux conditions (a) et (b')}\}$.

Si $T \in B(X)$ satisfait aux conditions (a), (b) et (b'), alors T est, par définition, un opérateur Fredholm.

On note par $\Phi(X)$ la classe des opérateurs Fredholm, sur l'espace X ; évidemment, on a $\Phi(X) = \Phi_+(X) \cap \Phi_-(X)$.

Maintenant, nous pouvons établir le théorème de caractérisation de la classe $\Phi_+(X)$ en utilisant la notion de mesure de non-compactité.

Théorème 1. Soit $T \in B(X)$; on a $T \in \Phi_+(X)$ si et seulement si il existe un nombre $\eta \in R_+^*$ de sorte que, quel que soit $E \in M(X)$ on ait la relation

$$(*) \quad q(E) \leq \eta q[T(E)].$$

Démonstration. Supposons que $T \in \Phi(X)$, alors il existe un opérateur T_0 dans la classe $B(X)$ tel qu'on a

$$(1) \quad TT_0 - I = K \in K(X) \text{ où } I \text{ est l'opérateur identité sur } X \text{ (v. [6]).}$$

Donc, pour chaque ensemble $E \in M(X)$, on obtient

$$(1') \quad q(E) = q[(TT_0 - K)(E)] \leq q[(TT_0)(E)] + q[K(E)];$$

où

$$(2) \quad q(E) \leq \eta q[T(E)] \quad \text{où} \quad \eta = \|T_0\|.$$

Si $T \in \Phi_+(X) - \Phi_-(X)$, alors, $\dim N(T) < +\infty$ et $\dim X /_{R(T)} = +\infty$.

En utilisant un résultat établi par B. Yood (v. [6]), on peut représenter l'opérateur T à l'aide de la relation

$$(3) \quad T = V_0 + K_0,$$

où $V_0 \in B(X)$ est un isomorphisme et $K_0 \in K(X)$.

Vu que V_0 est un isomorphisme, il existe une constante $c_0 \in R_+^*$ de manière que

$$(4) \quad \|(T - K_0)(x)\| \geq c_0 \|x\|, \quad x \in X.$$

Soient $E \in M(X)$, $\bar{E} \notin \mathcal{K}$ et $E \in R_+^*$ de sorte qu'il y ait un recouvrement fini réalisé, pour l'ensemble borné $(T - K_0)(E)$, par une famille finie d'ensembles $S(y_i, \epsilon)$, $y_i \in X$, $i = 1, 2, \dots, n$; donc

$$(5) \quad (T - K_0)(E) \subset \bigcup_{i=1}^n S(y_i, \epsilon).$$

Parce que l'opérateur $T - K_0$ est un isomorphisme, quel que soit $x \in E$, il existe un point y_i , $i = 1, n$, de façon que

$$(6) \quad \|(T - K_0)(x) - y_i\| < \epsilon.$$

Nous considérons la plus petite famille de sphères $\{S(y_i, \varepsilon)\}_{i=1}^n$ qui réalise un recouvrement pour l'ensemble $(T-K_0)(E)$; alors, quel que soit y_i , il existe un point $x_i \in E$ de sorte qu'on ait la relation (6).

Étant donné un point $y_i, i=1, 2, \dots, n$, il existe un point x_i dans E tel qu'on ait la relation

$$(6') \quad \|(T-K_0)(x_i) - y_i\| < \varepsilon.$$

Des relations (4), (5), et (6'), on déduit que quel que soit $x \in E$ il existe $k \in \overline{1, n}$ de sorte que

$$(7) \quad c_0 \|x - x_k\| \leq \|(T-K_0)(x - x_k)\| \leq \|(T-K_0)(x) - y_k\| + \|y_k - (T-K_0)(x_k)\| < 2\varepsilon,$$

donc

$$(7') \quad \|x - x_k\| < \frac{2\varepsilon}{c_0},$$

où $x \in E$ et x_k est associé à y_k par la condition précisée ci-dessus.

On en déduit que, si $\varepsilon \in R_+^*$ est tel qu'il existe un recouvrement fini pour $(T-K_0)(E)$ avec des sphères de rayon ε , alors on peut mettre en évidence un ensemble fini de points dans $E, \{x_1, x_2, \dots, x_n\}$, de sorte qu'on ait la relation

$$(8) \quad E \subset \bigcup_{i=1}^n S\left(x_i, \frac{2\varepsilon}{c_0}\right).$$

Par conséquent,

$$(9) \quad q(E) \leq \frac{2\varepsilon}{c_0}.$$

pour tout $\varepsilon \in R_+^*$ qui satisfait à la condition dont nous avons parlé plus haut. Donc, quel que soit $E \in M(X)$ on ait la relation

$$(10) \quad q(E) \leq \frac{2}{c_0} q[(T-K_0)(E)] \leq \frac{2}{c_0} q[T(E)].$$

Nous avons démontré que si $T \in \Phi_+(X)$, alors il existe $\eta \in R_+^*$, où $\eta = 2/c_0$, de sorte que $q(E) \leq \eta q[T(E)]$, pour tout $E \in M(X)$.

Maintenant, on prouve la suffisance de la condition énoncée. Supposons que pour l'opérateur $T \in B(X)$, il existe une constante $\eta \in R_+^*$ de manière que $q(E) \leq \eta q[T(E)]$ quel que soit $E \in M(X)$, où $\bar{E} \notin \mathcal{X}$.

Considérons, maintenant, la représentation de l'espace X donnée par la relation $X = N(T) \oplus W$, où W est un espace fermé dans X . Alors, $R(T) = T(W)$.

Pour démontrer que $R(T) = \overline{R(T)}$, c'est-à-dire $T(W) = \overline{T(W)}$ ou prouve que l'application $T|_W^{-1}$ est continue. En effet, si $T|_W^{-1}$ n'a pas la propriété énoncée, alors il existe $(x_n) \subset W$ de manière que $\|x_n\| = 1, n \in N$ et $\|T(x_n)\| \rightarrow 0$.

D'ici et de la relation (*) il résulte que $(\overline{x_n}) \in \mathcal{X}$ (v. [2]). Par conséquent, il existe $(x_{n_j}) \subset (x_n)$ et $x_0 \in W$ tel que $x_{n_j} \rightarrow x_0$; donc $T(x_{n_j}) \rightarrow T(x_0) = 0$.

Alors, $x_0 \in N(T)$ et $x_0 \in W$, c'est-à-dire $x_0 = 0$, ce qui est absurde parce que $\|x_{n_j}\| = 1$, $j \in N$.

Donc, T_W^{-1} est continue et sur la base d'un théorème connu de caractérisation pour les applications continues, il résulte que $T(W) = \overline{T(W)}$. Ensuite, on prouve que la $\dim N(T) < +\infty$. En effet, si $S_{N(T)}$ est la sphère unité dans $N(T)$ et si on suppose que la $\dim N(T) = \infty$, alors, $\overline{S_{N(T)}} \notin K$; donc $q(S_{N(T)}) > 0$ et $q[T(S_{N(T)})] > 0$.

Il en résulte que $\overline{T(S_{N(T)})} \notin \mathcal{K}$ ce qui est absurde parce que $T(S_{N(T)}) = \{0\}$.

On a démontré de cette manière que, dans les conditions données, $T \in \Phi_+(X)$ (v. la définition [3]).

Le Théorème 1 permet qu'on démontre aisément la proposition suivantes :

Proposition 3. Si $T_1, T_2 \in \Phi_+(X)$ alors $T_1 T_2 \in \Phi_+(X)$.

En effet, le Théorème 1 assure l'existence des $c_1, c_2 \in R_+^*$ de manière que

$$\begin{cases} q(E) \leq c_1 q[T_1(E)] \\ q(E) \leq c_2 q[T_2(E)] \end{cases} \quad \text{quel que soit } E \in M(X).$$

Il en résulte

$$q[(T_1 T_2)(E)] = q[T_1(T_2(E))] \geq \frac{1}{c_1} q[T_2(E)] \geq \frac{1}{c_1 c_2} q(E).$$

Donc, pour l'opérateur $T_1 T_2$, il existe une constante $c = c_1 c_2 \in R_+^*$ de sorte que, quel que soit $E \in M(X)$, on a $q(E) \leq c q[(T_1 T_2)(E)]$ et $T_1 T_2 \in \Phi_+(X)$.

Proposition 4. Si $T_1, T_2 \in B(X)$ et $T_2 T_1 \in \Phi_+(X)$, alors $T_2 \in \Phi_+(X)$.

Vu que $T_2 T_1 \in \Phi_+(X)$, il existe $c_1 \in R_+^*$ de manière que

$$(11) \quad q(E) \leq c_1 q[(T_2 T_1)(E)];$$

mais, $q[(T_2 T_1)(E)] \leq \|T_1\| q[T_2(E)]$.

Alors, on a

$$(12) \quad q(E) \leq c_1 \|T_1\| q[T_2(E)],$$

ou

$$(13) \quad q(E) \leq c_2 q[T_2(E)], \quad c_2 = c_1 \|T_1\|.$$

On a prouvé que la relation (13) est satisfaite pour chaque $E \in M(X)$ et tenant compte du Théorème 1, on déduit que $T_2 \in \Phi_+(X)$.

Proposition 5. Si $T \in \Phi_+(X)$ et $U \in K(X)$ alors $T + U \in \Phi_+(X)$.

En effet, on a $T(E) \subset (T + U)(E) - U(E)$ pour tout $E \in M(X)$. Donc,

$$(14) \quad q[T(E)] \leq q[(T + U)(E) - U(E)] \leq q[(T + U)(E)].$$

On obtient la relation (14) en tenant compte du fait que l'application est monotone et que l'opérateur U est compact.

Vu que $T \in \Phi_+(X)$, il existe $c \in R_+^*$ tel que

$$(15) \quad q(E) \leq c q[T(E)], \quad E \in M(X).$$

D'ici et de la relation (14) on trouve

$$(16) \quad q[(T+U)(E)] \geq q[T(E)] \geq \frac{1}{c} q(E).$$

C'est-à-dire $q(E) \leq cq[(T+U)(E)]$ quel que soit $E \in M(X)$. Ce qui démontre que $T+U \in \Phi_+(X)$.

Il faut remarquer qu'on peut prouver les propositions (3), (4), (5) par un autre procédé (v. par exemple [1]), mais, dans cette Note, elles sont démontrées en utilisant seulement la mesure de non-compactité.

À l'aide des propositions établies plus haut et en utilisant la liaison qui existe entre les classes d'opérateurs $\Phi_+(X)$, $\Phi_-(X^*)$; $\Phi_+(X^*)$, $\Phi_-(X)$, on démontre, aisément, la

Proposition 6. a) Si $T_1, T_2 \in \Phi_-(X)$ alors $T_1 T_2 \in \Phi_-(X)$.

b) Si $T_2 T_1 \in \Phi(X)$, alors $T_1 \in \Phi_-(X)$.

c) Si $T_2 T_1 \in \Phi(X)$, alors $T_2 \in \Phi_+(X)$ et $T_1 \in \Phi_-(X)$.

d) Si $T \in \Phi_-(X)$ et $U \in K(X)$ alors $T+U \in \Phi_-(X)$.

e) Si $T \in \Phi(X)$ et $U \in K(X)$, alors $T+U \in \Phi(X)$.

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STUDIUL CLASEI $\Phi_+(X)$ CU AJUTORUL MĂSURII NON-COMPACITĂȚII

(Rezumat)

Se introduce noțiunea de măsură de non-compactitate pentru operatori din $B(X)$, X spațiu Banach, și se pun în evidență proprietățile fundamentale ale aplicației menționate; aceste rezultate, împreună cu unele elemente de teoria generală a operatorilor Fredholm, permit să se stabilească o teoremă de caracterizare în limbajul măsurii de non-compactitate a claselor de operatori semi-Fredholm.

D'alte de la relaţia (14) se obţine

$$(10) \quad \varphi(T+U)(E) > \varphi(T)(E) > \frac{1}{2}\varphi(E)$$

C'est-à-dire $\varphi(E) < \varphi(T+U)(E)$, quel que soit $E \in M(X)$. Ce qui démontre que $T+U \in \Phi(X)$.

Il faut remarquer qu'on peut prouver les propositions (3), (4), (5) par un autre procédé (v. par exemple [1]), mais dans cette Note elles sont démontrées en utilisant seulement la mesure de non-compacité.

À l'aide des propositions établies plus haut et en utilisant la liaison qui existe entre les classes d'opérateurs $\Phi(X)$, $\Phi(X)$, $\Phi(X)$, $\Phi(X)$, on démontre aisément la

Proposition 6. Si $T, T_1 \in \Phi(X)$ alors $T_1 T = \Phi(X)$.

b) Si $T, T_1 \in \Phi(X)$, alors $T_1 \in \Phi(X)$.

c) Si $T_1 T = \Phi(X)$, alors $T \in \Phi(X)$ et $T_1 \in \Phi(X)$.

d) Si $T \in \Phi(X)$ et $U \in K(X)$ alors $T+U \in \Phi(X)$.

e) Si $T \in \Phi(X)$ et $U \in K(X)$, alors $T+U \in \Phi(X)$.

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STUDIUL CLASEI $\Phi(X)$ CU AJUTORUL MĂSURII NON-COMPACTĂȚII

(Romanian)

Se introduce noţiunea de măsură de non-compacitate pentru operatorii din $B(X, X)$ şi se demonstrează că aceasta este echivalentă cu măsură de non-compacitate în spaţiul Banach. Se demonstrează apoi că această măsură este echivalentă cu măsură de non-compacitate în spaţiul Banach. Se demonstrează apoi că această măsură este echivalentă cu măsură de non-compacitate în spaţiul Banach.

Secția I

MATEMATICĂ, MECANICĂ TEORETICĂ, FIZICĂ

C D, 517,5

APPLICATIONS DU THÉORÈME DE POINT FIXE DE BANACH

PAR

NICOLETA CIOBANU

Dans cette Note on considère des suites $x_{p+n} = f(x_n, x_{n+1}, \dots, x_{n+p-1})$, $n=1, 2, \dots$, où (X, d) est un espace métrique complet et $f: X^p \rightarrow X$, avec la propriété: ils existent $\alpha_1, \dots, \alpha_p \geq 0$, $0 < \sum_{i=1}^p \alpha_i = \alpha < 1$ tels que

$$d(f(x_1, \dots, x_p), f(x_2, \dots, x_{p+1})) \leq \alpha_1 d(x_1, x_2) + \alpha_2 d(x_2, x_3) + \dots + \alpha_p d(x_p, x_{p+1}),$$

et nous montrons qu'elles sont convergentes au point fixe unique $x \in X$ de la fonction $\tilde{f}: X \rightarrow X$, définie par $\tilde{f}(u) = f(u, u, \dots, u)$ pour tout $u \in X$, pour tout choix du point $(x_1, x_2, \dots, x_p) \in X^p$.

Le Théorème 1 n'est pas une application directe du théorème de point fixe de Banach, mais la méthode employée ici est une suggérée par celle utilisée dans le théorème de Banach.

Les Lemmes 1, 2 et 3 sont utiles pour la démonstration des théorèmes 1, 2, 3.

Les observations 2 et 5 utilisent le théorème de point fixe de Banach et l'observation 5 est nécessaire pendant la démonstration du Théorème 3 pour montrer que le point fixe de l'application P dépend continument des paramètres $z_1, \dots, z_q$.

L e m m e 1. Si (X, d) est un espace métrique, $f: X^p \rightarrow X$ et ils existent

$\alpha_1, \alpha_2, \dots, \alpha_p \in [0, 1]$ avec $0 < \sum_{i=1}^p \alpha_i = \alpha < 1$ tels que

$$(1) \quad \begin{aligned} & d(f(u_1, \dots, u_p), f(u_2, \dots, u_{p+1})) \leq \\ & \leq \alpha_1 d(u_1, u_2) + \alpha_2 d(u_2, u_3) + \dots + \alpha_p d(u_p, u_{p+1}) \end{aligned}$$

pour tous $u_1, \dots, u_{p+1} \in X$, alors

$$(2) \quad d(f(x, \dots, x), f(y, \dots, y)) \leq \alpha d(x, y) \text{ pour tous } x, y \in X.$$

Démonstration. $d(f(x, x, \dots, x), f(y, y, \dots, y)) \leq d(f(x, x, \dots, x), f(x, x, \dots, x, y)) + d(f(x, x, \dots, x, y), f(x, x, \dots, x, y, y)) + \dots + d(f(x, y, y, \dots, y),$

$$f(y, y, \dots, y) \leq \alpha_p d(x, y) + \alpha_{p-1} d(x, y) + \dots + \alpha_1 d(x, y) = \alpha d(x, y).$$

Observation 1. Le Théorème 1 montre que la condition (1) implique que la fonction $\tilde{f}: X \rightarrow X$ définie par $\tilde{f}(x) = f(x, x, \dots, x)$ est une contraction.

L e m m e 2. Si (X, d) est un espace métrique, $f: X^p \rightarrow X$ est une application avec la propriété (1) et $x \in X$ satisfait à l'équation $x = f(x, x, \dots, x)$, alors x est le point unique de X qui est la solution de cette équation.

D é m o n s t r a t i o n. S'il était encore un point $y \in X$, $y \neq x$ qui satisfait $y = f(y, y, \dots, y)$, alors $d(x, y) = d(f(x, x, \dots, x), f(y, y, \dots, y)) \leq \alpha d(x, y)$ et donc $(1 - \alpha) d(x, y) \leq 0$ ce qui implique $d(x, y) = 0$, c'est à dire $x = y$.

L e m m e 3. Si (X, d) est un espace métrique et $f: X^p \rightarrow X$ satisfait (1), alors il suit

$$(3) \quad d(f(u_1, u_2, \dots, u_p), f(u, u, \dots, u)) \leq 2\alpha(d(u_1, u) + d(u_2, u) + \dots + d(u_p, u)).$$

D é m o n s t r a t i o n. En employant l'inégalité du triangle et l'inégalité (1) on a

$$\begin{aligned} d(f(u_1, u_2, \dots, u_p), f(u, u, \dots, u)) &\leq d(f(u_1, \dots, u_p), f(u_2, u_3, \dots, u_p, u)) + \\ &+ d(f(u_2, u_3, \dots, u_p, u), f(u_3, u_4, \dots, u_p, u, u)) + \dots + d(f(u_p, u, u, \dots, u), \\ &f(u, u, \dots, u)) \leq \alpha_1 d(u_1, u_2) + \alpha_2 d(u_2, u_3) + \dots + \alpha_{p-1} d(u_{p-1}, u_p) + \alpha_1 d(u_p, u) + \\ &+ \alpha_1 d(u_2, u_3) + \alpha_3 d(u_3, u_4) + \dots + \alpha_{p-1} d(u_p, u) + \dots + \alpha_1 d(u_{p-1}, u) + \\ &+ \alpha_1 d(u_p, u) + \alpha_1 d(u_p, u) = \alpha_1 d(u_1, u_2) + (\alpha_1 + \alpha_2) d(u_2, u_3) + \\ &+ (\alpha_1 + \alpha_2 + \alpha_3) d(u_3, u_4) + \dots + (\alpha_1 + \alpha_2 + \dots + \alpha_p) d(u_p, u). \end{aligned}$$

En appliquant encore l'inégalité du triangle à chaque distance $d(u_i, u_{i+1}) \leq d(u_i, u) + d(u_{i+1}, u)$, $i = 1, \dots, p$, on obtient

$$\begin{aligned} d(f(u_1, \dots, u_p), f(u, u, \dots, u)) &\leq \alpha_1 d(u_1, u) + (2\alpha_1 + \alpha_2) d(u_2, u) + \\ &+ (2\alpha_1 + 2\alpha_2 + \alpha_3) d(u_3, u) + \dots + (2\alpha_1 + 2\alpha_2 + \dots + 2\alpha_{p-1} + \alpha_p) d(u_p, u). \end{aligned}$$

En majorant les coefficients des distances qui interviennent avec 2α , nous obtenons l'inégalité (3).

Observation 2. Si (X, d) est un espace métrique complet et ils existent

$$\alpha_1, \alpha_2, \dots, \alpha_p \in [0, 1), \quad 0 < \sum_{i=1}^p \alpha_i = \alpha < 1 \text{ tels que la condition (1) soit vraie,}$$

il suit que f est une contraction, f est continue sur X et, du théorème de Banach, il existe un point fixe unique $x \in X$ qui satisfait $x = f(x, x, \dots, x)$.

T h é o r è m e 1. Si (X, d) est un espace métrique complet et $f: X^p \rightarrow X$ qui a la propriété (1), alors la suite (X_{p+n}) définie par les relations

$$(4) \quad x_{p+n} = f(x_n, x_{n+1}, \dots, x_{n+p-1}) \quad (n = 1, 2, \dots)$$

est convergente au point fixe unique $x \in X$ de la fonction $\tilde{f}: X \rightarrow X$, définie par $\tilde{f}(u) = f(u, u, \dots, u)$, pour tout $u \in X$.

D é m o n s t r a t i o n. Nous montrons que dans les conditions du Lemme 1 quand $(x_1, x_2, \dots, x_p)$ est un point arbitraire dans X^p , sont vraies les inégalités

$$(5) \quad \begin{aligned} d(x_{pn+1}, x_{pn+2}) &\leq \alpha^n M_p, \\ d(x_{pn+2}, x_{pn+3}) &\leq \alpha^n M_p, \\ &\dots \dots \dots \end{aligned}$$

$$(6) \quad d(x_l, x_m) \leq p\alpha^{\left[\frac{l}{p}\right]} \frac{M_p}{1-\alpha} \quad \text{pour tous } l, m \in N, p < l < m \quad \text{et}$$

$$(7) \quad d(x_l, x) \leq p\alpha^{\left[\frac{l}{p}\right]} \frac{M_p}{1-\alpha}, \quad \text{où } M_p = \max \{d(u_1, u_2), d(u_2, u_3), \dots, \{d(u_p, u_{p+1})\}.$$

Pour montrer que (5) sont vraies nous utilisons la méthode de l'induction. Les relations (5) sont vraies pour $n=0$ car $d(x_1, x_2) \leq M_p, d(x_2, x_3) \leq M_p, \dots, d(x_p, x_{p+1}) \leq M_p$.

Nous supposons que les relations (5) sont vraies pour n et nous montrerons qu'elles sont vraies aussi pour $n+1$.

$$\begin{aligned} d(x_{p(n+1)+1}, x_{p(n+1)+2}) &= \\ &= d(f(x_{pn+1}, \dots, x_{p(n+1)}), f(x_{pn+2}, \dots, x_{p(n+1)+1})) \leq \\ &\leq \alpha_1 d(x_{pn+1}, x_{pn+2}) + \alpha_2 d(x_{pn+2}, x_{pn+3}) + \dots + \alpha_p d(x_{p(n+1)}, x_{p(n+1)+1}) \leq \\ &\leq \alpha_1 \alpha^n M_p + \alpha_2 \alpha^n M_p + \dots + \alpha_p \alpha^n M_p = \alpha^n M_p (\alpha_1 + \alpha_2 + \dots + \alpha_p) = \alpha^{n+1} M_p, \end{aligned}$$

$$\begin{aligned} d(x_{p(n+2)}, x_{p(n+2)+1}) &= \\ &= d(f(x_{p(n+1)}, x_{p(n+1)+1}, \dots, x_{p(n+2)-1}), f(x_{p(n+1)+1}, \dots, x_{p(n+2)})) \leq \\ &\leq \alpha_1 d(x_{p(n+1)}, x_{p(n+1)+1}) + \alpha_2 d(x_{p(n+1)+1}, x_{p(n+1)+2}) + \dots + \alpha_p d(x_{p(n+2)-1}, x_{p(n+2)}) \leq \\ &\leq \alpha_1 \alpha^n M_p + \alpha_2 \alpha^{n+1} M_p + \dots + \alpha_p \alpha^{n+1} M_p = \alpha^n M_p (\alpha_1 + \alpha_2 \alpha + \dots + \alpha_p \alpha) \leq \alpha^{n+1} M_p, \end{aligned}$$

donc les relations (5) sont vraies.

Soit $l, m \in N$ tels que $p < l < m$. En employant l'inégalité du triangle et les relations (5) nous obtenons $d(x_l, x_m) \leq p[\alpha^{\left[\frac{l}{p}\right]} + \alpha^{\left[\frac{l}{p}\right]+1} + \dots] M_p$, d'où suivent les relations (6), (7). La relation (7) suit du fait que la distance est une fonction continue et on peut calculer la limite pour $n \rightarrow \infty$.

Les relations (6) montrent que la suite (x_{p+n}) est une suite de Cauchy. L'espace métrique (X, d) étant complet, cette suite est convergente à un point $x \in X, x = \lim_{n \rightarrow \infty} x_{p+n}$. Pour tout $\varepsilon > 0$, en posant dans (3) $d(u_i, u) < \varepsilon / (2p\alpha), i=1, \dots, p$, on obtient que $d(f(u_1, \dots, u_p), f(u, u, \dots, u)) \leq 2\alpha p\varepsilon / (2p\alpha) = \varepsilon$, ce qui implique la continuité de la fonction $f: X^p \rightarrow X$ dans tout point $(u, u, \dots, u) \in X^p$.

La fonction f étant continue dans tout point $(x, x, \dots, x) \in X^p$, en faisant $n \rightarrow \infty$ dans (4), on trouve $x = f(x, x, \dots, x)$, c'est à dire que la suite (4) converge au point $x \in X$ qui satisfait l'équation $x = f(x, x, \dots, x)$.

Le point x est unique conformément au Lemme 2 et x est le point fixe de l'application $\tilde{f}: X \rightarrow X$, définie par $\tilde{f}(u) = f(u, u, \dots, u)$ pour tout $u \in X$.

Observation 3. Le Théorème 1 a été démontré d'une autre manière en [2] et il a été généralisé pour les espaces 2-métriques en [3].

Théorème 2. Soit X un espace de Banach et $f: X^p \rightarrow X$ une application pour laquelle

$$(8) \quad \begin{aligned} & \text{ils existent } p \text{ nombres réels } \alpha_1, \dots, \alpha_p \in [0, 1], \quad 0 < \sum_{i=1}^p \alpha_i = \alpha < 1 \text{ tels que} \\ & \|f(x_1, \dots, x_p) - f(x_2, \dots, x_{p+1})\| \leq \alpha_1 \|x_1 - x_2\| + \alpha_2 \|x_2 - x_3\| + \dots \\ & \dots + \alpha_p \|x_p - x_{p+1}\| \end{aligned}$$

pour tous $x_1, \dots, x_{p+1} \in X$. Alors l'application $\varphi: X \rightarrow X$, définie par $x \mapsto x - f(x, x, \dots, x)$, est surjective sur X .

Démonstration. Soit y un élément arbitraire de X et l'application $g: X^p \rightarrow X$, $(x_1, \dots, x_p) \mapsto f(x_1, x_2, \dots, x_p) + y$. Alors

$$\begin{aligned} \|g(x_1, \dots, x_p) - g(x_2, \dots, x_{p+1})\| &= \|f(x_1, \dots, x_p) - f(x_2, \dots, x_{p+1})\| \leq \\ &\leq \alpha_1 \|x_1 - x_2\| + \alpha_2 \|x_2 - x_3\| + \dots + \alpha_p \|x_p - x_{p+1}\|. \end{aligned}$$

L'application $\tilde{g}$, satisfaisant les conditions du Lemme 1, est une contraction. X , étant un espace de Banach, est aussi un espace métrique complet et alors l'application $\tilde{g}: X \rightarrow X$, $\tilde{g}(x) = g(x, x, \dots, x)$ a un point fixe unique $u \in X$ conformément au théorème de Banach, donc $g(u, u, \dots, u) = u$ où $f(u, u, \dots, u) + y = u$. On obtient ainsi que pour tout $y \in X$, il existe $u \in X$, tel que $u - f(u, u, \dots, u) = y$.

Observation 4. Si $p=1$ dans le Théorème 2, on obtient le Théorème 3.2.3. de I.A. Rus [1, p. 31].

Définition 1. Soit $(z_1, z_2, \dots, z_q)$ un point fixé de $Z_1 \times Z_2 \times \dots \times Z_q$, où $X_1, Z_1, \dots, Z_q$ sont $q+1$ ensembles. Nous définissons l'application $f_z: X \rightarrow X$ par la relation $f_z(x, x, \dots, x) = f(x, x, \dots, x, z_1, \dots, z_q)$, où $z = (z_1, \dots, z_q)$ et notons $\tilde{f}_z: X \rightarrow X$ l'application définie par $\tilde{f}_z(x) = f(x, x, \dots, x, z_1, z_2, \dots, z_q)$.

Observation 5. Si $z_1 \in Z_1, z_2 \in Z_2, \dots, z_q \in Z_q$ et (X, d) est un espace métrique complet, alors, conformément au Lemme 1, f_z , qui satisfait à la condition (1) et à l'observation 2, est une contraction et donc, du théorème de point fixe de Banach, elle a un point fixe unique $x_z^* \in X$, c'est à dire $f_z(x_z^*, x_z^*, \dots, x_z^*) = x_z^*$.

Définition 2. Notons $P: Z_1 \times Z_2 \times \dots \times Z_q \rightarrow X$ l'application définie par $(z_1, z_2, \dots, z_q) \mapsto x_z^*$, c'est à dire $P(z_1, z_2, \dots, z_q) = x_z^*$.

Définition 3. Notons f_x la fonction $f_x: Z_1 \times Z_2 \times \dots \times Z_q \rightarrow X$ définie par $(z_1, z_2, \dots, z_q) \mapsto f(x, x, \dots, x, z_1, z_2, \dots, z_q)$, c'est à dire $f_x(z_1, z_2, \dots, z_q) = f(x, x, \dots, x, z_1, z_2, \dots, z_q)$.

Théorème 3. Si

(i) (X, d) est un espace métrique complet, $Z_1, \dots, Z_q$ sont q espaces topologiques de Hausdorff et $f: X^p \times Z_1 \times Z_2 \times \dots \times Z_q \rightarrow X$ tels qu'ils existent $\alpha_1, \dots,$

$\alpha_p \in [0, 1], \quad 0 < \sum_{i=1}^p \alpha_i = \alpha < 1$ avec la propriété

$$d(f(x_1, \dots, x_p, z_1, \dots, z_q), f(x_2, \dots, x_{p+1}, z_1, \dots, z_q)) \leq$$

$$\leq \alpha_1 d(x_1, x_2) + \alpha_2 d(x_2, x_3) + \dots + \alpha_p d(x_p, x_{p+1})$$

pour tous $x_1, \dots, x_{p+1} \in X$ et $z_1 \in Z_1, \dots, z_q \in Z_q$ et

(ii) l'application f_x est continue pour tout $x \in X$, alors l'application P est univoque et continue.

Démonstration. De l'observation 5 et de la définition 2 il suit que $P(z_1, \dots, z_q) = x_z^*$ qui est le point fixe unique de la fonction $\tilde{f}_z$ et il satisfait l'équation $f(x_z^*, x_z^*, \dots, x_z^*, z_1, \dots, z_q) = x_z^*$. Donc la fonction P est univoque.

Pour montrer la continuité de P nous observons que $P(z_1, \dots, z_q) = x_z^*$ et $\tilde{f}_z(P(z_1, \dots, z_q)) = \tilde{f}_z(x_z^*) = f(x_z^*, x_z^*, \dots, x_z^*, z_1, \dots, z_q) = x_z^* = P(z_1, \dots, z_q)$. Donc $P(z_1, \dots, z_q) = \tilde{f}_z(P(z_1, \dots, z_q))$. Soit $z^0 = (z_1^0, \dots, z_q^0) \in Z_1 \times \dots \times Z_q$. Alors on a

$$\begin{aligned} d(P(z_1, \dots, z_q), P(z_1^0, \dots, z_q^0)) &= d(\tilde{f}_z(P(z_1, \dots, z_q)), \tilde{f}_z(P(z_1^0, \dots, z_q^0))) \leq \\ &\leq d(\tilde{f}_z(P(z_1, \dots, z_q)), \tilde{f}_z(P(z_1^0, \dots, z_q^0))) + d(\tilde{f}_z(P(z_1^0, \dots, z_q^0)), \tilde{f}_z(P(z_1^0, \dots, z_q^0))) \leq \\ &\leq \alpha d(P(z_1, \dots, z_q), P(z_1^0, \dots, z_q^0)) + d(\tilde{f}_z(P(z_1^0, \dots, z_q^0)), \tilde{f}_z(P(z_1^0, \dots, z_q^0))) \end{aligned}$$

conformément au Lemme 1 et l'observation 5.

Donc

$$d(P(z_1, \dots, z_q), P(z_1^0, \dots, z_q^0)) \leq \frac{1}{1-\alpha} d(\tilde{f}_z(P(z_1^0, \dots, z_q^0)), \tilde{f}_z(P(z_1^0, \dots, z_q^0))).$$

Mais

$$\tilde{f}_z(P(z_1^0, \dots, z_q^0)) = \tilde{f}_z(x_{z^0}^*) = f(x_{z^0}^*, \dots, x_{z^0}^*, z_1, \dots, z_q) = f_{x_{z^0}^*}(z_1, \dots, z_q)$$

et

$$f_{x_{z^0}^*}(P(z_1^0, \dots, z_q^0)) = f_{x_{z^0}^*}(z_1^0, \dots, z_q^0).$$

Donc

$$d(P(z_1, \dots, z_q), P(z_1^0, \dots, z_q^0)) \leq \frac{1}{1-\alpha} d(f_{x_{z^0}^*}(z_1, \dots, z_q), f_{x_{z^0}^*}(z_1^0, \dots, z_q^0)).$$

f_x est une fonction continue pour tout $x \in X$ et donc pour $x_{z^0}^*$, ce que implique la distance du second membre de la dernière relation converge à 0 quand $(z_1, \dots, z_q) \rightarrow (z_1^0, \dots, z_q^0)$. Le fait que les espaces topologiques $Z_1, \dots, Z_q$ sont séparés Hausdorff assure l'unicité de la limite $\lim_{(z_1, \dots, z_q) \rightarrow (z_1^0, \dots, z_q^0)} P(z_1, \dots, z_q)$ et

l'égalité $\lim_{(z_1, \dots, z_q) \rightarrow (z_1^0, \dots, z_q^0)} P(z_1, \dots, z_q) = P(z_1^0, \dots, z_q^0)$. Donc l'application P est continue.

Observation 6. Le Théorème 4 suit du Théorème 3 pour $q=1$, $p=1$.

Théorème 4. (I. A. Rus [1, p. 32]). Soit (X, d) un espace métrique complet, Z un espace topologique de Hausdorff et $f: X \times Z \rightarrow X$ qui ont les propriétés:

(i) il existe $\alpha \in (0, 1)$ tel que $d(f(x_1, z), f(x_2, z)) \leq \alpha d(x_1, x_2)$ pour tous $x_1, x_2 \in X$ et tous $z \in Z$;

(ii) l'application f_x est continue.

Alors l'application P est univoque et continue.

Observation 7. La suite $x_{p+n}=f(x_n, \dots, x_{n+p-1}, z_1, \dots, z_q)$, dans les conditions du Théorème 3, est convergente et $\lim_{n \rightarrow \infty} x_{p+n} = x_z^* = f(x_z^*, x_z^*, \dots, x_z^*, z_1, \dots, z_q)$.

Le Théorème 3 implique que cette limite est continue par rapport à tous les paramètres $z_1 \in Z_1, z_2 \in Z_2, \dots, z_q \in Z_q$.

Observation 8. On peut aussi énoncer les résultats précédents pour des fonctions

$$f: X^{p_1} Z_1 X^{p_2} Z_2 \times \dots \times X^{p_q} Z_q \rightarrow X, \text{ où } \sum_{i=1}^q p_i = p.$$

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APLICAȚII ALE TEOREMEI DE PUNCT FIX A LUI BANACH

(Rezumat)

Se consideră șirurile de forma $x_{p+n}=f(x_n, x_{n+1}, \dots, x_{n+p-1})$, unde (X, d) este un spațiu metric complet și $f: X^p \rightarrow X$ cu proprietatea că există $\alpha_1, \dots, \alpha_p \in [0, 1)$ cu $0 < \sum_{i=1}^p \alpha_i = \alpha < 1$ astfel că $d(f(x_1, \dots, x_p), f(x_2, \dots, x_{p+1})) \leq \alpha_1 d(x_1, x_2) + \alpha_2 d(x_2, x_3) + \dots + \alpha_p d(x_p, x_{p+1})$ și se arată că ele converg la punctul fix $x \in X$ unic al aplicației $\tilde{f}: X \rightarrow X$, definită prin $\tilde{f}(u) = f(u, u, \dots, u)$, $u \in X$ independent de alegerea punctului inițial $(x_1, x_2, \dots, x_p) \in X^p$.

Teorema 1 nu este o aplicație directă a teoremei de punct fix a lui Banach, dar metoda de demonstrație folosită aici este una sugerată de cea utilizată în teorema lui Banach. Lemele 1, 2 și 3 sînt utile pentru demonstrațiile teoremelor 1, 2 și 3.

Observațiile 2 și 5 folosesc teorema de punct fix a lui Banach. Observația 5 este necesară în demonstrația teoremei 3 pentru a arăta că punctul fix al aplicației f depinde continuu de parametrii $z_1, z_2, \dots, z_q$ sau că punctul limită al șirului $x_{n+p} = (x_n, \dots, x_{n+p-1}, z_1, \dots, z_q)$ depinde continuu de $z_1, z_2, \dots, z_q$.

F-CONNECTIONS ON A CR-SUBMANIFOLD

BY

AUREL BEJANCU

0. Introduction. Let N be an almost Hermitian manifold of complex dimension n . Denote by J the almost complex structure on N . Suppose M is a Riemannian manifold of real dimension m immersed in N . Denote by the same symbol g both metrics on M and N . For each $x \in M$ denote by $T_x M$ and $T_x M^\perp$ respectively the tangent and the normal space to M at x .

Definition [2]. The submanifold M is called a CR-submanifold of N if the following conditions are satisfied:

- (i) there exists a differentiable distribution $D: x \rightarrow D_x \subset T_x M$ on M which is holomorphic, i.e. $J(D_x) = D_x$ for each $x \in M$, and
- (ii) the complementary orthogonal distribution $D^\perp: x \rightarrow D_x^\perp \subset T_x M$ is anti-invariant, i.e. $J D_x^\perp \subset T_x M^\perp$ for each $x \in M$.

We denote by P and Q the projection morphisms of the tangent bundle TM to D and $D^\perp$ respectively. By means of these projectors we define the f -structure F on M (see [7] for the definition of an f -structure) and a normal bundle valued 1-form u on M by

$$(0.1) \quad FX = JPX$$

and respectively

$$(0.2) \quad uX = JQX,$$

for each vector field X tangent to M . The fundamental properties of F and u are given in [4].

Next, we define a 2-form G on M by

$$(0.3) \quad G(X, Y) = g(X, FY),$$

for all X, Y tangent to M . The Nijenhuis tensor of F is given by

$$(0.4) \quad N_F(X, Y) = [FX, FY] - F[X, FY] - F[FX, Y] + F^2[X, Y].$$

Then, the f -structure F is integrable if and only if N_F vanishes identically on M .

A linear connection ∇ on M is said to be an F -connection if the f -structure F is covariant constant with respect to this connection, that is we have

$$(0.5) \quad \nabla_X F = 0$$

for each vector field X tangent to M .

The main purpose of the present paper is to obtain all the F -connections on the CR -submanifold M (Theorem 1.2). Then, by means of suitable conditions we prove the existence of an F -connection without torsion on a CR -submanifold immersed in a Hermitian manifold (Theorem 2.1). Finally, we obtain a necessary and sufficient condition for the Levi-Civita connection on M in order to be an F -connection (Theorem 2.2).

1. F -connections on a CR -submanifold. Let M be a differentiable manifold endowed with two complementary distributions D and $D^\perp$. This means that $TM = D \oplus D^\perp$ where TM is the tangent bundle to M . Denote by P and Q the projection morphisms of TM to D and $D^\perp$ respectively. The modules of all differentiable sections of vector bundles TM , D and $D^\perp$ are denoted by $\Gamma(TM)$, $\Gamma(D)$ and $\Gamma(D^\perp)$.

Throughout the paper, by „differentiable“ we mean differentiable of class C^∞ .

Definition. The distribution D and $D^\perp$ are called parallel with respect to a linear connection ∇ on M if we have

$$(1.1) \quad Q(\nabla_X Y) = 0, \quad \forall X \in \Gamma(TM), \quad Y \in \Gamma(D) \quad \text{and respectively}$$

$$(1.2) \quad P(\nabla_X Z) = 0, \quad \forall X \in \Gamma(TM), \quad Z \in \Gamma(D^\perp).$$

The existence of a linear connection with respect to which both distributions D and $D^\perp$ are parallel has been proved by A. G. Walker [5]. We give here by an invariant form all such linear connections.

Theorem 1.1. *All the linear connections with respect to which both distributions D and $D^\perp$ are parallel, are given by*

$$(1.3) \quad \nabla_X Y = P\dot{\nabla}_X PY + Q\dot{\nabla}_X QY + PS(X, PY) + QS(X, QY), \\ \forall X, Y \in \Gamma(TM),$$

where $\dot{\nabla}$ and S are respectively an arbitrary linear connection on M and an arbitrary tensor field of type $(1, 2)$ on M .

Proof. We take an arbitrary linear connection $\dot{\nabla}$ on M . Then, any linear connection ∇ on M is given by

$$(1.4) \quad \nabla_X Y = \dot{\nabla}_X Y + S(X, Y),$$

where S is an arbitrary tensor field of type $(1, 2)$ on M . From the conditions (1.1) and (1.2) follows

$$(1.5) \quad QS(X, Y) + Q\dot{\nabla}_X Y = 0, \quad \forall X \in \Gamma(TM), \quad Y \in \Gamma(D) \quad \text{and}$$

$$(1.6) \quad PS(X, Z) + P\dot{\nabla}_X Z = 0, \quad \forall X \in \Gamma(TM), \quad Z \in \Gamma(D^\perp).$$

Thus, by using (1.5) and (1.6) in (1.4) we obtain (1.3).

Now, suppose M is a CR -submanifold of an almost Hermitian manifold N . Then, we have

Theorem 1.2. *All the F -connections on the CR -submanifold M are given by*

$$(1.7) \quad \nabla_X Y = P\dot{\nabla}_X PY + Q\dot{\nabla}_X QY + \frac{1}{2} [(\dot{\nabla}_X F)FY + PK(X, PY) - FK(X, FY)] + QS(X, QY),$$

where $\dot{\nabla}$ is a linear connection with respect to which both distributions D and $D^\perp$ are parallel and K, S are arbitrary tensor fields of type $(1, 2)$ on M .

Proof. Suppose ∇ is an F -connection on M . Then, we put

$$(1.8) \quad \nabla_X Y = \dot{\nabla}_X Y + S(X, Y), \quad \forall X, Y \in \Gamma(TM),$$

where $\dot{\nabla}$ is a linear connection on M with respect to which both distributions D and $D^\perp$ are parallel and S is an arbitrary tensor field of type $(1, 2)$ on M .

The existence of $\dot{\nabla}$ has been proved by Theorem 1.1.

Since ∇ has to satisfy (0.5) we have

$$(1.9) \quad FS(X, Y) - S(X, FY) = (\dot{\nabla}_X F)Y, \quad \forall X, Y \in \Gamma(TM).$$

Substituting Y by FY in (1.9) we obtain

$$(1.10) \quad FS(X, FY) + S(X, PY) = (\dot{\nabla}_X F)FY.$$

Denote by S_X the tensor field of type $(1, 1)$ on M defined by

$$(1.11) \quad S_X(Y) = S(X, PY), \quad \forall Y \in \Gamma(TM).$$

Then, (1.10) becomes

$$(1.12) \quad S_X + FS_X F = (\dot{\nabla}_X F)F,$$

since $PF = F$. From (1.11) follows

$$(1.13) \quad S_X(Z) = 0, \quad \forall Z \in \Gamma(D^\perp).$$

On the other hand, by using (1.12) and taking account that D is parallel with respect to $\dot{\nabla}$ we obtain

$$(1.14) \quad Q(S_X(Y)) = 0, \quad \forall Y \in \Gamma(D).$$

Next, denote by $\Gamma_1^1(TM)$ the real vector space of all tensor fields of type $(1, 1)$ on M . Also, we consider the vector subspace $\Gamma_1^1(D)$ of $\Gamma_1^1(TM)$ defined by

$$(1.15) \quad \Gamma_1^1(D) = \{K \in \Gamma_1^1(TM) ; KQY = QKPY = 0 ; \forall Y \in \Gamma(TM)\}.$$

We define two linear operators Φ and Ψ on $\Gamma_1^1(D)$ by

$$(1.16) \quad \Phi K = \frac{1}{2}(K + FKF)$$

and respectively

$$(1.17) \quad \Psi K = \frac{1}{2}(K - FKF).$$

By a straightforward computation, using (1.15)–(1.17) we obtain that Φ and Ψ are complementary projectors on $\Gamma_1^1(D)$, that is, we have

$$(1.18) \quad \Phi^2 = \Phi; \quad \Psi^2 = \Psi; \quad \Phi\Psi = \Psi\Phi = 0; \quad \Phi + \Psi = I,$$

where I is the identity morphism on $\Gamma_1^1(D)$. By means of (1.13) and (1.14) we see that $S_x \in \Gamma_1^1(D)$. Hence (1.12) becomes

$$(1.19) \quad \Phi(S_x) = \frac{1}{2}(\dot{\nabla}_x F)F.$$

We have further

$$(1.20) \quad \Psi\Phi(S_x) = \frac{1}{2}[(\dot{\nabla}_x F)F + F(\dot{\nabla}_x F)P] = 0,$$

since $\dot{\nabla}_x P = 0$ on M . Hence the equation (1.19) has the solution

$$(1.21) \quad S_x = \frac{1}{2}[(\dot{\nabla}_x F)F + K_x - FK_x F],$$

where K_x is an arbitrary element of $\Gamma_1^1(D)$. Thus we have

$$(1.22) \quad S(X, PY) = \frac{1}{2}[(\dot{\nabla}_x F)FY + K(X, Y) - FK(X, FY)],$$

$$\forall X, Y \in \Gamma(TM),$$

where K is a tensor field of type (1,2) on M which satisfies

$$(1.23) \quad K(X, QY) = QK(X, PY) = 0.$$

Now, from (1.8) follows

$$(1.24) \quad (\nabla_x F)QY = (\dot{\nabla}_x F)QY - FS(X, QY), \quad \forall X, Y \in \Gamma(TM).$$

Since ∇ has to be an F -connection and $D\perp$ is parallel with respect to $\dot{\nabla}$, from (1.24) we have

$$(1.25) \quad PS(X, QY) = 0, \quad \forall X, Y \in \Gamma(TM).$$

By using (1.14), (1.22), (1.23) and (1.25) in (1.8) we obtain (1.7).

Finally, it is a simple verification that conditions (1.14), (1.22), (1.23) and (1.25) are sufficient for ∇ in order to define an F -connection on M . The proof of the theorem is complete.

From Theorems 1.1. and 1.2. we obtain

Theorem 1.3. *Both distributions on a CR-submanifold M of an almost Hermitian manifold N are parallel with respect to any F -connection on M .*

Theorem 1.4. *Let M be a CR-submanifold of an almost Hermitian manifold N . If the Levi-Civita connection on M is an F -connection then M is*

locally a Riemannian direct product $M^{\mathbf{T}} \times M^{\perp}$ where $M^{\mathbf{T}}$ is an almost Hermitian submanifold of N and $M^{\perp}$ is a totally real submanifold of N .

Remark. The Theorem 1.4 has been proven by K. Yano, M. Kon and the present author [1] for generic CR-submanifolds of a Kaehler manifold.

2. Particular F -connections on a CR-submanifold. In this paragraph we are looking for particular F -connections on M . First we shall prove the existence of a less torsion F -connection on a particular CR-submanifold.

By direct computation, taking account of (0.4) and (0.5) we obtain

$$(2.1) \quad N_F(X, Y) = P \overset{(1)}{T}(X, Y) + F \overset{(1)}{T}(FX, Y) + F \overset{(1)}{T}(X, FY) - \overset{(1)}{T}(FX, FY),$$

where $\overset{(1)}{T}$ is the torsion tensor of an F -connection $\overset{(1)}{\nabla}$ on M .

Thus, we have

Proposition 2.1. *Let M be a CR-submanifold of an almost Hermitian manifold N . If there exists a less torsion F -connection on M , then the f -structure F is integrable.*

Now, we can prove

Theorem 2.1. *Let M be a CR-submanifold of a Hermitian manifold N . If the f -structure F is integrable, then there exists an F -connection without torsion on M .*

Proof. First we shall prove the existence of an F -connection

$\overset{(1)}{\nabla}$ whose torsion tensor $\overset{(1)}{T}$ verifies

$$(2.2) \quad P \overset{(1)}{T}(X, QY) = 0, \quad \forall X, Y \in \Gamma(TM).$$

Since N_F vanishes identically on M , from (2.1) follows $P \overset{(1)}{T}(QX, QY) = 0$. Thus, in order to obtain (2.2) it is sufficient to prove

$$(2.3) \quad P \overset{(1)}{T}(PX, QY) = 0.$$

By Theorems 3.2. and 3.3 from [3] we obtain that both distributions D and $D^{\perp}$ are integrable on M . Then, by Theorem 4.8 ([6], p. 250) there exists a symmetric linear connection $\overset{(0)}{\nabla}$ with respect to which both distributions D and $D^{\perp}$ are parallel.

Next, we take an F -connection $\overset{(1)}{\nabla}$ given by (1.7) where $\overset{(1)}{\nabla}$ is the linear connection chosen above. By a direct computation we have

$$(2.4) \quad P \overset{(1)}{T}(PX, QY) = \frac{1}{2} [P \overset{(1)}{\nabla}_{QX} P X + F \overset{(1)}{\nabla}_{QX} F P X - \\ - P K(QY, P X) + F K(QY, F X)].$$

From $N_F(PX, QY) = 0$ follows

$$(2.5) \quad P[PX, QY] + F[FPX, QY] = 0$$

which implies

$$(2.6) \quad P\dot{\nabla}_{QY}PX + F\dot{\nabla}_{QY}FPX = 0.$$

We choose $K=0$ on M and by (2.4) and (2.6) follows (2.3).

Now, from (2.1) follows

$$(2.7) \quad Q\overset{(1)}{T}(X, Y) = 0 \quad \text{for all } X, Y \in \Gamma(D).$$

On the other hand, by using (1.7) we obtain

$$(2.8) \quad Q\overset{(1)}{T}(X, Y) = Q\{\dot{\nabla}_X Y - [X, Y] + \frac{1}{2} \dot{\nabla}_Y X + S(X, Y)\},$$

$$\forall X \in \Gamma(D), \quad Y \in \Gamma(D^\perp);$$

$$(2.9) \quad Q\overset{(1)}{T}(X, Y) = Q\{\dot{T}(X, Y) + S(X, Y) - S(Y, X)\}, \quad \forall X, Y \in \Gamma(D^\perp).$$

We take $S=0$ on M and by (2.7)–(2.9) we obtain

$$(2.10) \quad Q\overset{(1)}{T}(X, Y) = 0, \quad \forall X, Y \in \Gamma(TM),$$

since $\dot{\nabla}$ is a symmetric connection with respect to which D and $D^\perp$ are parallel.

For each $X \in \Gamma(TM)$ we define the tensor field $\overset{(1)}{T}_X \in \Gamma_1^1(TM)$ by

$$(2.11) \quad \overset{(1)}{T}_X(Y) = \overset{(1)}{T}(X, Y), \quad \forall Y \in \Gamma(TM).$$

By (2.2) it is easily seen that $P\overset{(1)}{T}_X \in \Gamma_1^1(D)$ for each $X \in \Gamma(TM)$. We have further

$$(2.12) \quad F\overset{(1)}{T}(X, FY) = F(P\overset{(1)}{T}_X)FY = P\overset{(1)}{T}(X, Y) - 2\Psi(P\overset{(1)}{T}_X)(Y).$$

In a similar way we obtain

$$(2.13) \quad F\overset{(1)}{T}(FX, Y) = P\overset{(1)}{T}(X, Y) + 2\Psi(P\overset{(1)}{T}_Y)(X),$$

$$(2.14) \quad P\overset{(1)}{T}(FX, FY) = -P\overset{(1)}{T}(X, Y) - 2\Psi(P\overset{(1)}{T}_Y)(X) + 2\Psi(F\overset{(1)}{T}_{FX})(Y),$$

Taking account of the skew-symmetry of $\overset{(1)}{T}$, (2.14) becomes

$$(2.15) \quad P\overset{(1)}{T}(FX, FY) = -P\overset{(1)}{T}(X, Y) + \Psi(P\overset{(1)}{T}_X)(Y) - \Psi(P\overset{(1)}{T}_Y)(X) + \\ + \Psi(F\overset{(1)}{T}_{FX})(Y) - \Psi(F\overset{(1)}{T}_{FY})(X).$$

By using (2.12), (2.13) and (2.15) in (2.1) we obtain

$$(2.16) \quad 4P\overset{(1)}{T}(X, Y) + 3\Psi(P\overset{(1)}{T}_Y)(X) - \Psi(F\overset{(1)}{T}_{FY})(X) - 3\Psi(P\overset{(1)}{T}_X)(Y) + \\ + \Psi(F\overset{(1)}{T}_{FX})(Y) = 0,$$

for all vector fields X, Y tangent to M .

Finally, we take the linear connection ∇ given by

$$(2.17) \quad \nabla_X Y = \overset{(1)}{\nabla}_X Y + \frac{1}{4} \Psi(3P\overset{(1)}{T}_Y - F\overset{(1)}{T}_{FY})(X).$$

It is not difficult to see that ∇ is an F -connection on M . Moreover, by using (2.10), (2.16) and (2.17) we obtain that ∇ has no torsion. The proof is complete.

From now on we suppose that M is a CR -submanifold of a Kaehler manifold N . We shall denote by ∇ the Levi-Civita connection on M .

First we recall [4]

Lemma 2.1. *The covariant derivative of F with respect to the Levi-Civita connection ∇ on M is given by*

$$(2.18) \quad \begin{aligned} 2g((\nabla_X F)Y, Z) = & 3dG(X, FY, FZ) - 3dG(X, Y, Z) + g(N_F(Y, Z), FX) + \\ & + 2g(du(FY, Z) - du(FZ, Y), uX) + 2g(du(FY, X), uZ) - \\ & - 2g(du(FZ, X), uY), \end{aligned}$$

for all vector fields X, Y, Z tangent to M .

Next, for each differentiable normal section ξ to M we put

$$(2.19) \quad J\xi = B\xi + C\xi,$$

where $B\xi \in \Gamma(D^\perp)$ and $C\xi \in \Gamma(TM^\perp)$. Then we have [4]

$$(2.20) \quad (\nabla_X F)Y = Bh(X, Y) + A_{uY}X,$$

$$(2.21) \quad (\nabla_X u)Y = Ch(X, Y) - h(X, FY),$$

where h is the second fundamental form of M and A_{uY} is the fundamental tensor of Weingarten with respect to the normal section uY .

Now we can prove

Theorem 2.2. *Let M be a CR -submanifold of a Kaehler manifold N . Then, the following conditions are equivalent to each other:*

1. *The Levi-Civita connection ∇ on M is an F -connection on M .*
2. *The f -structure F is integrable, the 2-form G is closed ($dG=0$) and the 1-form u satisfies*

$$(2.22) \quad Bdu(X, Y) = 0, \quad \forall X \in \Gamma(D) \quad \text{and} \quad Y \in \Gamma(D^\perp).$$

Proof. Suppose condition 1 is accomplished. By Proposition 2.1 follows $N_F=0$ on M , that is F is integrable on M . Further, by using (2.21) we obtain

$$(2.23) \quad Bdu(X, Y) = \frac{1}{2} Bh(JX, Y), \quad \forall X \in \Gamma(D), \quad Y \in \Gamma(D^\perp).$$

But, from (2.20) taking account that ∇ is an F -connection we have $Bh(JX, Y)=0$. Thus, (2.22) follows from (2.23).

On the other hand, by Theorem 3.1. from [3] follows the integrability of the holomorphic distribution. Then, from (2.18) we obtain

$$(2.24) \quad dG(X, FY, FZ) = dG(X, Y, Z), \quad \forall X, Y, Z \in \Gamma(TM).$$

Substituting Y by X in (2.21) we have $dG(X, FX, FZ) = 0$ which implies

$$(2.25) \quad dG(X, FY, FZ) + dG(Y, FX, FZ) = 0.$$

Also, from (2.24) we obtain

$$(2.26) \quad dG(X, Y, Z) = 0, \quad \forall Y \in \Gamma(D\perp), \quad X, Z \in \Gamma(TM).$$

By using (2.24) – (2.26) and the skew-symmetry of dG we get

$$(2.27) \quad dG(X, FY, FZ) = -dG(X, Y, Z), \quad \forall X, Y, Z \in \Gamma(TM).$$

Thus, by (2.24) and (2.27) follows that G is a closed 2-form on M .

Now, suppose the condition 2 of the theorem is satisfied. The holomorphic distribution is involutive since F is integrable. Thus by using (2.22) we have

$$(2.28) \quad Bdu(FX, Y) = 0, \quad \forall X, Y \in \Gamma(TM).$$

Then, from (2.18) follows that the Levi-Civita connection is an F -connection. This completes the proof of the theorem.

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F-CONEXIUNI PE O CR-SUBVARIETATE

(Rezumat)

Se introduce noțiunea de F -conexiune pe o CR -subvarietate și apoi se găsesc toate F -conexiunile; se demonstrează existența unei F -conexiuni fără torsiune pe o CR -subvarietate imersată într-o varietate hermitiană.

CIRCULAȚIE ALTERNATĂ ÎN GRAFURI EVANTAI ȘI ÎN GRAFURI EVANTAI ÎNCHISE

DE
VIOREL GHIUȘ

Fie un graf evantai [3] (fig. 1) unde P_i și R_i sînt muchii sau drumuri simple și $n \geq 2$ (săgețile grafului indică sensul circulației alternate care va fi introdusă mai jos).

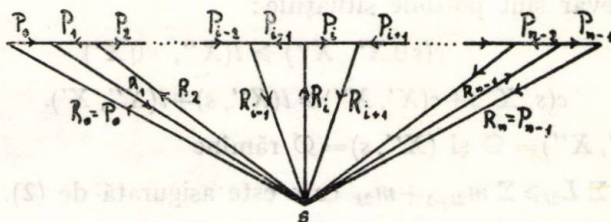


Fig. 1

Vrem să studiem circulația în acest graf neorientat, știind că

$$(1) \quad \begin{cases} 0 \leq l_i \leq F(R_i) \leq L_i & i=0, 1, \dots, n \\ 0 \leq m_i \leq F(P_i) \leq M_i, & i=1, 2, \dots, n-1 \end{cases}$$

Considerăm relațiile

$$(2) \quad \max(m_{i-1} + m_i, l_i) \leq \min(M_{i-1} + M_i, L_i) \quad i=1, 2, \dots, n-1;$$

$$(3) \quad \max\left(\sum_{k=0}^{[n/2]} l_{2k}, \sum_{k=0}^{[n-1/2]} l_{2k+1}\right) \leq \min\left(\sum_{k=0}^{[n/2]} L_{2k}, \sum_{k=0}^{[n-1/2]} L_{2k+1}\right).$$

Teorema 1. *Condiția necesară și suficientă ca într-un graf evantai să existe o circulație satisfăcînd (1) de forma*

$$(4) \quad F(R_i) = F(P_{i-1}) + F(P_i) \quad \text{pentru } i=1, 2, \dots, n-1,$$

și

$$(5) \quad \sum_{k=0}^{[n/2]} F(R_{2k}) = \sum_{k=0}^{[n-1/2]} F(R_{2k+1}),$$

unde sensul circulației pe P_{2k} este contrar sensului circulației pe P_{2k+1} , este ca să fie îndeplinite condițiile (2) și (3).

Observație. Sensul circulației pe P_i implică sensul circulației pe R_i și anume: sensul circulației pe R_{2k} este contrar sensului circulației pe R_{2k+1} (o astfel de circulație o vom numi circulație alternată).

Necesitatea. Din (4) și (1) avem

$$l_i \leq F(R_i) \leq L_i,$$

$$m_{i-1} + m_i \leq F(P_{i-1}) + F(P_i) \leq M_{i-1} + M_i \text{ pentru } i=1, 2, \dots, n-1,$$

de unde rezultă (2).

Din (5) și (1) avem

$$\Sigma F(R_{2k}) = \Sigma F(R_{2k+1}),$$

$$\Sigma l_{2k} \leq \Sigma F(R_{2k}) \leq \Sigma L_{2k},$$

$$\Sigma l_{2k+1} \leq \Sigma F(R_{2k+1}) \leq \Sigma L_{2k+1},$$

de unde rezultă (3).

Suficiența. Notăm că X' nodurile inițiale pentru R_{2k+1} și cu X'' nodurile finale pentru R_{2k} . Graful evantai poate fi orientat conform circulației alternate specificate, deoarece condițiile (2) și (3) asigură îndeplinirea condițiilor (3.3) din teorema 3.1 [1].

Într-adevăr sînt posibile situațiile:

$$a) \quad c(s \cup X', X'') \geq l(X'', s \cup X')$$

$$\text{sau} \quad c(s, X'') + c(X', X'') \geq l(X'', s) + l(X'', X').$$

Deoarece $(X', X'') = \emptyset$ și $(X'', s) = \emptyset$ rămîne

$$\Sigma L_{2k} \geq \Sigma m_{2k-1} + m_{2k} \text{ care este asigurată de (2).}$$

$$b) \quad c(X', s \cup X'') \geq l(s \cup X'', X') \text{ implică}$$

$$\Sigma L_{2k+1} \geq \Sigma m_{2k} + m_{2k+1} \text{ care este asigurată de (2).}$$

$$c) \quad c(s \cup X'', X') \geq l(X', s \cup X'') \text{ implică}$$

$$\Sigma M_{2k} + M_{2k+1} \geq \Sigma l_{2k+1} \text{ care este asigurată de (2).}$$

$$d) \quad c(X'', s \cup X') \geq l(s \cup X', X'') \text{ implică}$$

$$\Sigma M_{2k-1} + M_{2k} \geq \Sigma l_{2k} \text{ care este asigurată de (2).}$$

$$e) \quad c(s, X' \cup X'') \geq l(X' \cup X'', s) \text{ implică}$$

$$\Sigma L_{2k} \geq \Sigma l_{2k+1} \text{ care este asigurată de (3).}$$

$$f) \quad c(X' \cup X'', s) \geq l(s, X' \cup X'') \text{ implică}$$

$$\Sigma L_{2k+1} \geq \Sigma l_{2k} \text{ care este asigurată de (3).}$$

Condițiile (3.2) din teorema 3.1 [1] asigură condițiile (1), iar condițiile (3.1) asigură (4) și (5). Tot din [1] se poate folosi algoritmul de construcție a acestei circulații alternate.

Dacă avem satisfăcută în plus condiția

$$(6) \quad m_{i-1} + m_i \geq l_i \quad \text{pentru} \quad i=1, 2, \dots, n-1,$$

(2) se reduce la

$$(2') \quad m_{i-1} + m_i \leq L_i$$

și rezultă următorul

Corolar. În ipotezele (3), (6) și (2'), graful evantai respectiv admite o circulație de forma (1), alternată de tipul (4), (5) definită astfel:

$$F(P_i) = m_i \quad \text{pentru} \quad i=0, 1, \dots, n-1.$$

Rezultă că $F(R_i) = m_{i-1} + m_i$ pentru $i=1, 2, \dots, n-1$ și se verifică banal (4) precum și (5).

Să considerăm următorul graf evantai închis (fig. 2) unde P_i și R_i sînt muchii sau drumuri simple pentru $i=1, 2, \dots, 2n$ (săgețile grafului indică sensul circulației alternate care va fi introdusă ulterior).

Vrem să studiem o circulație de forma (1) în acest graf neorientat în condițiile (2) și (3) pentru $i=1, 2, \dots, 2n$.

În mod similar se poate demonstra

Teorema 2. Condiția necesară și suficientă ca într-un graf evantai închis să existe o circulație satisfăcînd (1) de forma (4) și (5) pentru $i=1, 2, \dots, 2n$, unde sensul circulației pe P_{2k} este contrar sensului circulației pe P_{2k+1} , este ca să fie îndeplinite condițiile (2) și (3).

Observație. Sensul circulației pe P_i implică sensul circulației pe R_i și anume: sensul circulației pe R_{2k} este contrar sensului circulației pe R_{2k+1} (se definește o circulație alternată); în plus numărul muchiilor (drumurile simple) P_i și R_i trebuie să fie par.

Dacă este satisfăcută condiția (6) pentru $i=1, 2, \dots, 2n$ are loc următorul

Corolar. Într-un graf evantai închis există o circulație (1), alternată, de forma (4) și (5) definită prin

$$F(P_i) = m_i, \quad i=1, 2, \dots, 2n,$$

iar $F(R_i) = m_{i-1} + m_i$ pentru $i=1, 2, \dots, 2n$.

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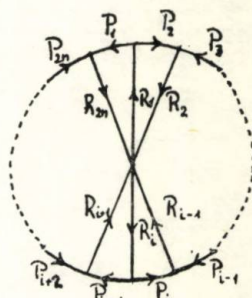


Fig. 2

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LA CIRCULATION ALTERNÉE DANS LES GRAPHES ÉVENTAILS ET DANS LES GRAPHES ÉVENTAILS FERMÉS

(Résumé)

On définit la notion de circulation alternée et on étudie le problème de la circulation alternée dans tels graphes. Le problème de la circulation alternée dans les graphes éventails et les graphes éventails fermés peut être posé et résolu à l'aide de la circulation dans des graphes orientés.

On obtient des conditions nécessaires et suffisantes pour l'existence de cette catégorie de graphes.



En modifiant les conditions de circulation alternée, on peut démontrer que dans un graphe éventail fermé, il existe une circulation alternée si et seulement si les conditions de circulation alternée sont satisfaites.

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On obtient des conditions nécessaires et suffisantes pour l'existence de cette catégorie de graphes.

ASUPRA PUNCTELOR DE ARTICULAȚIE ALE UNUI GRAF ORIENTAT FINIT

DE

DĂNUȚ MARCU

În lucrarea de față se stabilește o teoremă de existență și o teoremă de caracterizare a punctelor de articulație ale unui graf orientat finit, utilizând conceptul de *bloc* definit în [2].

Vom considera un graf orientat finit (v. [1]), $G = \langle \mathcal{N}, \mathcal{A} \rangle$, unde $\mathcal{N} = \{n_1, n_2, \dots, n_p\}$ este mulțimea de noduri, iar $\mathcal{A} = \{a_1, a_2, \dots, a_q\}$, mulțimea de arce.

Fie $n_0, m_0 \in \mathcal{N}$ două noduri ale grafului G . Numim lanț (v. [2]) (notat $\gamma[n_0, m_0]$) de la n_0 la m_0 , o secvență de noduri de forma $(n_{\alpha_1}, n_{\alpha_2}, \dots, n_{\alpha_r})$ pentru care avem:

$$n_0 = n_{\alpha_1},$$

$$m_0 = n_{\alpha_r},$$

$$\langle n_{\alpha_{j-1}}, n_{\alpha_j} \rangle \in \mathcal{A} \quad \text{sau} \quad \langle n_{\alpha_j}, n_{\alpha_{j-1}} \rangle \in \mathcal{A},$$

pentru orice $j = 2, 3, \dots, r$.

Notăm prin $\mathcal{N}(\gamma[n_0, m_0])$ mulțimea de noduri a lanțului $\gamma[n_0, m_0]$. Despre două noduri ale grafului G spunem că sînt *reciproc conexe* (v. [2]) dacă și numai dacă există $\gamma[n_0, m_0]$ (notăm aceasta prin $n_0 \sim m_0$). Două noduri n_0, m_0 sînt *adiacente* dacă $\langle n_0, m_0 \rangle \in \mathcal{A}$ sau $\langle m_0, n_0 \rangle \in \mathcal{A}$.

Numim subgraf (v. [1]) al unui graf orientat finit $G = \langle \mathcal{N}, \mathcal{A} \rangle$, un graf de forma $S = \langle \tilde{\mathcal{N}}, \tilde{\mathcal{A}} \rangle$, unde $\tilde{\mathcal{N}} \subset \mathcal{N}$, $\tilde{\mathcal{A}} = (\tilde{\mathcal{N}} \times \tilde{\mathcal{N}}) \cap \mathcal{A}$. Despre un graf spunem că este *conex* (v. [2]), dacă și numai dacă pentru oricare două noduri n_0 și m avem $n_0 \sim m_0$.

Fie $G = \langle \mathcal{N}, \mathcal{A} \rangle$ un graf orientat finit conex. Un nod $n_0 \in \mathcal{N}$ se numește *punct de articulație* al grafului G (vezi [2]) dacă și numai dacă subgraful $\langle \mathcal{N} - \{n_0\}, [(\mathcal{N} - \{n_0\}) \times (\mathcal{N} - \{n_0\})] \cap \mathcal{A} \rangle$ nu este conex. Fie $\mathcal{B} = \langle \mathcal{B}, (\mathcal{B} \times \mathcal{B}) \cap \mathcal{A} \rangle$ un subgraf al lui G . Spunem că $\mathcal{B}$ este un *bloc* (v. [2]) dacă și numai dacă se verifică simultan condițiile:

(α) $\mathcal{B}$ este conex;

(β) nu există nici un nod $n_0 \in \mathcal{B}$ astfel încît graful $\langle \mathcal{B} - \{n_0\}, [(\mathcal{B} - \{n_0\}) \times (\mathcal{B} - \{n_0\})] \cap \mathcal{A} \rangle$ să nu fie conex;

(γ) B este maximal cu proprietățile (α) și (β).

Lema 1. Dacă $B_1 = \langle \mathcal{B}_1, (\mathcal{B}_1 \times \mathcal{B}_1) \cap \mathcal{A} \rangle$ și $B_2 = \langle \mathcal{B}_2, (\mathcal{B}_2 \times \mathcal{B}_2) \cap \mathcal{A} \rangle$ sînt două blocuri distincte ale unui graf orientat finit $G = \langle \mathcal{K}, \mathcal{A} \rangle$ atunci, $|B_1 \cap B_2| \leq 1$.

Demonstrație. Presupunem că există $n_0, m_0 \in \mathcal{K}$ astfel încît $n_0, m_0 \in B_1 \cap B_2$. Întrucît B_1 și B_2 verifică condițiile (α) și (β), rezultă că există $\gamma[n_0, m_0]$ și prin urmare graful $\langle \mathcal{B}_1 \cup \mathcal{B}_2, [(\mathcal{B}_1 \cup \mathcal{B}_2) \times (\mathcal{B}_1 \cup \mathcal{B}_2)] \cap \mathcal{A} \rangle$ verifică condițiile (α) și (β), ceea ce înseamnă că B_1 și B_2 nu sînt blocuri. Deci $|B_1 \cap B_2| \leq 1$.

Lema 2. Dacă $G = \langle \mathcal{K}, \mathcal{A} \rangle$ este un graf orientat finit și $a_0 \in \mathcal{A}$ un arc ales arbitrar, atunci, există unic un bloc $B_0 = \langle \mathcal{B}_0, (\mathcal{B}_0 \times \mathcal{B}_0) \cap \mathcal{A} \rangle$ astfel încît $a_0 \in (\mathcal{B}_0 \times \mathcal{B}_0) \cap \mathcal{A}$.

Demonstrație. Fie $a_0 = \langle n_0, m_0 \rangle \in \mathcal{A}$, arcul ales arbitrar. Evident, graful $\tilde{Q} = \langle \{n_0, m_0\}, (\{n_0, m_0\} \times \{n_0, m_0\}) \cap \mathcal{A} \rangle$ este conex și nu conține nici un punct de articulație.

Prin urmare, sau $\tilde{Q}$ este un bloc, sau este conținut într-un bloc. Fie $B_0 = \langle \mathcal{B}_0, (\mathcal{B}_0 \times \mathcal{B}_0) \cap \mathcal{A} \rangle$ un bloc, astfel încît $\tilde{Q} \subseteq B_0$. Să arătăm că B_0 este unic. Într-adevăr, presupunînd că mai există un bloc $B_1 = \langle \mathcal{B}_1, (\mathcal{B}_1 \times \mathcal{B}_1) \cap \mathcal{A} \rangle$ astfel încît $\tilde{Q} \subseteq B_1$, rezultă că $\{n_0, m_0\} \in \mathcal{B}_0 \cap \mathcal{B}_1$ și deci $|\mathcal{B}_0 \cap \mathcal{B}_1| \geq 2$; contradicție cu lema 1 și prin urmare B_0 este unic cu proprietatea cerută.

Lema 3. Dacă n_0 este un punct de articulație al unui graf orientat finit conex $G = \langle \mathcal{K}, \mathcal{A} \rangle$, atunci, există $n_\alpha^{[0]}, n_\beta^{[0]} \in \mathcal{K}$, astfel încît n_0 este adiacent nodurilor $n_\alpha^{[0]}$ și $n_\beta^{[0]}$, și oricare ar fi $\gamma[n_\alpha^{[0]}, n_\beta^{[0]}]$, rezultă $n_0 \in \mathcal{K}(\gamma[n_\alpha^{[0]}, n_\beta^{[0]}])$.

Demonstrație. Fie $\tilde{G} = \langle \mathcal{K} - \{n_0\}, [(\mathcal{K} - \{n_0\}) \times (\mathcal{K} - \{n_0\})] \cap \mathcal{A} \rangle$, $\mathcal{C}_0$ o componentă conexă a lui $\tilde{G}$ și $\bar{\mathcal{C}}_0 = (\mathcal{K} - \{n_0\}) \setminus \mathcal{C}_0$.

Întrucît graful G este conex, n_0 este punct de articulație, iar $\mathcal{C}_0$ și $\bar{\mathcal{C}}_0$ formează o partiție a mulțimii $\mathcal{K} - \{n_0\}$, există $n_\alpha^{[0]} \in \mathcal{C}_0$ și $n_\beta^{[0]} \in \bar{\mathcal{C}}_0$ adiacente nodului n_0 .

Întrucît G este conex și nici un lanț nu leagă nodurile $n_\alpha^{[0]}, n_\beta^{[0]}$ în $\tilde{G}$, înseamnă că pentru orice $\gamma[n_\alpha^{[0]}, n_\beta^{[0]}]$ în G , avem $n_0 \in \mathcal{K}(\gamma[n_\alpha^{[0]}, n_\beta^{[0]}])$.

Teorema 1. Fie $G = \langle \mathcal{K}, \mathcal{A} \rangle$ un graf orientat finit, conex și $m_1, m_2 \in \mathcal{K}$, astfel încît există două blocuri $B_1 = \langle \mathcal{B}_1, (\mathcal{B}_1 \times \mathcal{B}_2) \cap \mathcal{A} \rangle$, $B_2 = \langle \mathcal{B}_2, (\mathcal{B}_2 \times \mathcal{B}_2) \cap \mathcal{A} \rangle$ pentru care $m_1 \in \mathcal{B}_1$, $m_2 \in \mathcal{B}_2$. În aceste condiții există $n_0 \in \mathcal{K}$, punct de articulație, astfel încît oricare ar fi $\gamma[m_1, m_2]$, rezultă $n_0 \in \mathcal{K}(\gamma[m_1, m_2])$.

Demonstrație. Vom arăta că există un nod $n_0 \in \mathcal{K}$ astfel încît oricare ar fi lanțul $\gamma[m_1, m_2]$, avem $n_0 \in \mathcal{K}(\gamma[m_1, m_2])$ (astfel de lanțuri există, dat fiind faptul că G este conex). Într-adevăr, dacă nu ar exista nici un nod cu proprietatea enunțată mai sus, atunci, graful $\langle \mathcal{B}_1 \cup \mathcal{B}_2, [(\mathcal{B}_1 \cup \mathcal{B}_2) \times (\mathcal{B}_1 \cup \mathcal{B}_2)] \cap \mathcal{A} \rangle$ verifică condițiile (α) și (β) și prin urmare B_1 și B_2 nu sînt blocuri.

Este limpede, dat fiind faptul că orice lanț $\gamma[m_1, m_2]$ trece prin n_0 și G este conex, că nodul n_0 este punct de articulație al grafului G .

Teorema 2. Fie $G = \langle \mathcal{K}, \mathcal{A} \rangle$ un graf orientat finit, conex. Un nod $n_0 \in \mathcal{K}$ este punct de articulație al lui G , dacă și numai dacă există două

blocuri $B_1 = \langle \mathcal{B}_1, (\mathcal{B}_1 \times \mathcal{B}_1) \cap \mathcal{A} \rangle$ și $B_2 = \langle \mathcal{B}_2, (\mathcal{B}_2 \times \mathcal{B}_2) \cap \mathcal{A} \rangle$ astfel încât $n_0 \in \mathcal{B}_1 \cap \mathcal{B}_2$.

Demonstrație. Fie $n_0 \in \mathcal{N}$ un punct de articulație al grafului G . Conform lemei 3 există două noduri $n_\alpha^{[0]}$ și $n_\beta^{[0]} \in \mathcal{N}$, adiacente lui n_0 astfel încât oricare ar fi $\gamma[n_\alpha^{[0]}, n_\beta^{[0]}]$ avem $n_0 \in \mathcal{N}(\gamma[n_\alpha^{[0]}, n_\beta^{[0]}])$. Fie $a_\alpha^{[0]}$, $a_\beta^{[0]}$ arcele formate de n_0 cu $n_\alpha^{[0]}$, respectiv cu $n_\beta^{[0]}$. Este limpede că arcele $a_\alpha^{[0]}$ și $a_\beta^{[0]}$ nu pot aparține simultan unuia și aceluiași bloc, deoarece prin eliminarea nodului n_0 nodurile $n_\alpha^{[0]}$ și $n_\beta^{[0]}$ nu mai sînt reciproc conexe (orice lanț $\gamma[n_\alpha^{[0]}, n_\beta^{[0]}]$ trece prin n_0 , și prin urmare condiția (β) nu mai este verificată.

Deci, există două blocuri $B_1 = \langle \mathcal{B}_1, (\mathcal{B}_1 \times \mathcal{B}_1) \cap \mathcal{A} \rangle$ și $B_2 = \langle \mathcal{B}_2, (\mathcal{B}_2 \times \mathcal{B}_2) \cap \mathcal{A} \rangle$, astfel încât $a_\alpha^{[0]} \in (\mathcal{B}_1 \times \mathcal{B}_1) \cap \mathcal{A}$, iar $a_\beta^{[0]} \in (\mathcal{B}_2 \times \mathcal{B}_2) \cap \mathcal{A}$. Dar dacă $a_\alpha^{[0]} \in (\mathcal{B}_1 \times \mathcal{B}_1) \cap \mathcal{A}$ iar $a_\beta^{[0]} \in (\mathcal{B}_2 \times \mathcal{B}_2) \cap \mathcal{A}$, înseamnă că $n_0 \in \mathcal{B}_1$ și $n_0 \in \mathcal{B}_2$, adică $n_0 \in \mathcal{B}_1 \cap \mathcal{B}_2$.

Să presupunem acum că există două blocuri B_1 și B_2 astfel încât $n_0 \in \mathcal{B}_1 \cap \mathcal{B}_2$ și să arătăm că n_0 este punct de articulație. Fie $n_1^{[0]} \in \mathcal{B}_1$ și $n_2^{[0]} \in \mathcal{B}_2$ două noduri alese arbitrar în B_1 și B_2 . Dacă n_0 nu ar fi punct de articulație al grafului G , atunci $\langle \mathcal{N} - \{n_0\}, [(\mathcal{N} - \{n_0\}) \times (\mathcal{N} - \{n_0\})] \cap \mathcal{A} \rangle$, este conex și prin urmare există un lanț $\gamma[n_1^{[0]}, n_2^{[0]}]$ astfel încât $n_0 \notin \mathcal{N}(\gamma[n_1^{[0]}, n_2^{[0]}])$. Deci, graful $\langle \mathcal{B}_1 \cup \mathcal{B}_2, [(\mathcal{B}_1 \cup \mathcal{B}_2) \times (\mathcal{B}_1 \cup \mathcal{B}_2)] \cap \mathcal{A} \rangle$ verifică condițiile (α) și (β) ceea ce contrazice ipoteza că B_1 și B_2 sînt blocuri. Deci n_0 este punct de articulație al grafului G .

Cololar. Orice nod al unui graf orientat finit conex care nu este punct de articulație aparține unui și numai unui singur bloc.

Demonstrația corolarului este evidentă, avînd în vedere teorema 2, lema 1 și lema 2.

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ON THE JOINT POINTS OF AN ORIENTED FINITE GRAPH

(Summary)

In this paper we give two theorems concerning the joint points of an oriented finite graph, using the blocks from [2].

blocați $B_1 = \langle B_1, (B_1 \times B_1) \cap \text{act} \rangle$ și $B_2 = \langle B_2, (B_2 \times B_2) \cap \text{act} \rangle$ astfel încât $a_0 \in B_1 \cup B_2$.

Dacă a_0 este a_1 și a_2 sunt puncte de articulație ale grafului G . Considerăm două blocuri B_1 și B_2 astfel încât $a_1 \in B_1$ și $a_2 \in B_2$. Dacă a_1 și a_2 sunt puncte de articulație ale grafului G , atunci B_1 și B_2 sunt blocuri. Este limpede că B_1 și B_2 sunt blocuri și pot aparține simultan unui și aceluși bloc de articulație prin eliminarea nodului a_0 deoarece a_0 și a_1 nu sunt reciproc conexe prin a_2 și a_0 și a_2 și a_1 sunt reciproc conexe prin a_0 .

Dacă există două blocuri B_1 și B_2 astfel încât $a_1 \in B_1$ și $a_2 \in B_2$ și a_1 și a_2 sunt puncte de articulație ale grafului G , atunci B_1 și B_2 sunt blocuri. Este limpede că B_1 și B_2 sunt blocuri și pot aparține simultan unui și aceluși bloc de articulație prin eliminarea nodului a_0 deoarece a_0 și a_1 nu sunt reciproc conexe prin a_2 și a_0 și a_2 și a_1 sunt reciproc conexe prin a_0 .

Se presupune acum că există două blocuri B_1 și B_2 astfel încât $a_1 \in B_1$ și $a_2 \in B_2$ și a_1 și a_2 sunt puncte de articulație ale grafului G . Dacă a_1 și a_2 sunt puncte de articulație ale grafului G , atunci B_1 și B_2 sunt blocuri. Este limpede că B_1 și B_2 sunt blocuri și pot aparține simultan unui și aceluși bloc de articulație prin eliminarea nodului a_0 deoarece a_0 și a_1 nu sunt reciproc conexe prin a_2 și a_0 și a_2 și a_1 sunt reciproc conexe prin a_0 .

Concluzie. Dacă a_1 și a_2 sunt puncte de articulație ale grafului G , atunci B_1 și B_2 sunt blocuri. Este limpede că B_1 și B_2 sunt blocuri și pot aparține simultan unui și aceluși bloc de articulație prin eliminarea nodului a_0 deoarece a_0 și a_1 nu sunt reciproc conexe prin a_2 și a_0 și a_2 și a_1 sunt reciproc conexe prin a_0 .

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Prin urmare, dacă a_1 și a_2 sunt puncte de articulație ale grafului G , atunci B_1 și B_2 sunt blocuri. Este limpede că B_1 și B_2 sunt blocuri și pot aparține simultan unui și aceluși bloc de articulație prin eliminarea nodului a_0 deoarece a_0 și a_1 nu sunt reciproc conexe prin a_2 și a_0 și a_2 și a_1 sunt reciproc conexe prin a_0 .

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ON THE JOINT POINTS OF AN ORIENTED FINITE GRAPH

(Summary)

In this paper we give two theorems concerning the joint points of an oriented finite graph, using the ideas from [2].

LIMITA ȘI CONTINUITATEA ANALITICĂ A MULTIFUNCȚIILOR (II)

DE

AL. SABADIȘ

1. Proprietăți ale limitelor analitice de multifuncții. În cele ce urmează, ne vom referi la multifuncția (E, Y, F) , $E \subset X$ și la punctul $x_0 \in X$, care este punct de acumulare pentru E .

Propoziția 1. Dacă există limita analitică a multifuncției (E, Y, F) în punctul x_0 , ea este unică.

Demonstrație. Dacă ar avea loc $(a) - \lim_{x \rightarrow x_0} F(x) = A \in \mathcal{P}(Y)$ și $(a) - \lim_{x \rightarrow x_0} F(x) = B \in \mathcal{P}(Y)$, $A \neq B$, atunci pentru un șir $(x_n) \subset E$, $x_n \neq x_0$, $x_n \rightarrow x_0$, am avea $F(x_n) \xrightarrow{(a)} A$ și $F(x_n) \xrightarrow{(a)} B$, ceea ce este imposibil. Rezultă $A = B$.

Propoziția 2. Fie multifuncția $(E, \mathcal{C}, F)$, unde $\mathcal{C}$ este un clan din $\mathcal{P}(E)$, iar $(\mathcal{C}, [0, +\infty), m)$, o măsură. Dacă pentru multifuncția F există $(a) - \lim_{x \rightarrow x_0} F(x) = A \in \mathcal{C}$, atunci există limita funcției numerice $m \circ F$ în punctul x_0 și are loc relația

$$\lim_{x \rightarrow x_0} (m \circ F)(x) = m((a) - \lim_{x \rightarrow x_0} F(x)) = m(A).$$

Demonstrație. Pentru orice șir $(x_n) \subset E$, $x_n \neq x_0$, $x_n \rightarrow x_0$, are loc $F(x_n) \xrightarrow{(a)} A \in \mathcal{C}$. În baza teoremei 5 din [1], avem $m(F(x_n)) \xrightarrow{n \rightarrow \infty} m(A)$, ceea ce demonstrează propoziția.

Propoziția 3. Fie $\mathcal{C} \subset \mathcal{P}(Y)$, un clan Cauchy, multifuncția $(E, \mathcal{C}, F)$ și măsura $(\mathcal{C}, [0, +\infty), m)$. Dacă pentru orice șir $(x_n) \subset E$, $x_n \neq x_0$, $x_n \rightarrow x_0$, $(F(x_n))$ este un șir Cauchy de mulțimi, atunci funcția numerică $m \circ F$ are limită în x_0 .

Demonstrație. În clanul Cauchy $\mathcal{C}$, șirul Cauchy $(F(x_n))$ este analitic convergent în $\mathcal{C}$, pentru orice șir $(x_n) \subset E$, $x_n \neq x_0$, $x_n \rightarrow x_0$. Deci, există $(a) - \lim_{x \rightarrow x_0} F(x) = A \in \mathcal{C}$ și $\lim_{x \rightarrow x_0} (m \circ F)(x) = m(A)$.

Propoziția 4. Multifuncția (E, Y, F) are limita analitică $A \in \mathcal{P}(Y)$ în punctul x_0 , dacă și numai dacă multifuncțiile $(E, Y, F - A)$ și $(E, Y, A - F)$ au în x_0 limita analitică vidă.

Demonstrație. Avem

$$\begin{aligned}
& [(a) - \lim_{x \rightarrow x_0} F(x) = A \in \mathcal{P}(Y)] \Leftrightarrow [\forall (x_n) \subset E, x_n \neq x_0, x_n \rightarrow x_0 \Rightarrow \\
& \Rightarrow F(x_n) \xrightarrow{(a)} A] \Leftrightarrow [\forall (x_n) \subset E, x_n \neq x_0, x_n \rightarrow x_0 \Rightarrow F(x_n) \Delta A = \\
& = (F(x_n) - A) \cup (A - F(x_n)) \subseteq D_n, D_n \downarrow \Phi] \Leftrightarrow [\forall (x_n) \subset E, x_n \neq x_0, x_n \rightarrow \\
& \rightarrow x_0 \Rightarrow F(x_n) - A \subseteq D_n \text{ și } A - F(x_n) \subseteq D_n, D_n \downarrow \Phi] \Leftrightarrow [\forall (x_n) \subset E, \\
& x_n \neq x_0, x_n \rightarrow x_0 \Rightarrow F(x_n) - A \xrightarrow{(a)} \text{și } A - F(x_n) \xrightarrow{(a)} \Phi] \Leftrightarrow [\forall (x_n) \subset E, \\
& x_n \neq x_0, x_n \rightarrow x_0 \Rightarrow (F - A)(x_n) \xrightarrow{(a)} \Phi \text{ și } (A - F)(x_n) \xrightarrow{(a)} \Phi].
\end{aligned}$$

Propoziția 5. Fie multifuncțiile (E, Y, F) și (E, Y, G) . Presupunem că G are limită analitică vidă în x_0 . Dacă pentru orice șir $(x_n) \subset E$, $x_n \neq x_0$, $x_n \rightarrow x_0$, există un indice $n_0 \in \mathbb{N}$, astfel încît $n > n_0$ atrage $F(x_n) \subseteq G(x_n)$, atunci are loc relația

$$(a) - \lim_{x \rightarrow x_0} F(x) = \Phi.$$

Demonstrație. Oricare ar fi șirul $(x_n) \subset E$, $x_n \neq x_0$, $x_n \rightarrow x_0$, cu $n > n_0$, avem $F(x_n) \subseteq G(x_n) \subseteq D_n$, $D_n \downarrow \Phi$. Ori, de aici, rezultă că $(a) - \lim_{x \rightarrow x_0} F(x) = \Phi$.

Propoziția 6. Fie multifuncția (E, Y, F) cu limită analitică în x_0 . Dacă pentru orice șir $(x_n) \subset E$, $x_n \neq x_0$, $x_n \rightarrow x_0$, există un indice $n_0 \in \mathbb{N}$, astfel încît $n > n_0$ atrage $F(x_n) \subseteq B$, $B \in \mathcal{P}(Y)$, atunci $(a) - \lim_{x \rightarrow x_0} F(x) \subseteq B$.

Demonstrație. În baza definiției 2.2. și teoremei 2.2 din [4], rezultă că $(b) - \lim_{x \rightarrow x_0} F(x) = (a) - \lim_{x \rightarrow x_0} F(x) \subseteq B$.

Propoziția 7. Considerăm multifuncțiile (E, Y, F) , (E, Y, G) și (E, Y, H) , cu limită analitică în x_0 . Dacă pentru orice șir $(x_n) \subset E$, $x_n \neq x_0$, $x_n \rightarrow x_0$ există un indice $n_0 \in \mathbb{N}$, astfel încît $n > n_0$ atrage $F(x_n) \subseteq G(x_n) \subseteq H(x_n)$, atunci are loc relația

$$(a) - \lim_{x \rightarrow x_0} F(x) \subseteq (a) - \lim_{x \rightarrow x_0} G(x) \subseteq (a) - \lim_{x \rightarrow x_0} H(x).$$

Demonstrație. În baza definiției 2.2. și teoremei 2.2. din [4], rezultă că $(a) - \lim_{x \rightarrow x_0} F(x) \subseteq (a) - \lim_{x \rightarrow x_0} G(x) \subseteq (a) - \lim_{x \rightarrow x_0} H(x)$.

Observație. Dacă, în particular, $(a) - \lim_{x \rightarrow x_0} F(x) = (a) - \lim_{x \rightarrow x_0} H(x) = A \in \mathcal{P}(Y)$ atunci $(a) - \lim_{x \rightarrow x_0} G(x) = A$.

Propoziția 8. Dacă multifuncția (E, Y, F) are limită analitică în punctul x_0 și dacă există $A \in \mathcal{P}(Y)$, cu proprietatea $A \subset (a) - \lim_{x \rightarrow x_0} F(x)$, atunci pentru orice șir $(x_n) \subset E$, $x_n \neq x_0$, $x_n \rightarrow x_0$, există un indice $n_0 \in \mathbb{N}$, astfel încît $n > n_0$ atrage $x \in F(x_n)$, oricare ar fi $x \in A$.

Demonstrație. Dacă $B = (a) - \lim_{x \rightarrow x_0} F(x)$, atunci pentru orice șir $(x_n) \subset E$, $x_n \neq x_0$, $x_n \rightarrow x_0$, are loc $F(x_n) \xrightarrow{(a)} B$. B este formată din puncte ce aparțin unui număr infinit de mulțimi $F(x_n)$ și lipsesc numai dintr-un număr finit de mulțimi $F(x_n)$. Așadar, din $A \subset B$, rezultă că orice punct $x \in A$ aparține unei infinități de mulțimi de forma $F(x_n)$ și lipsește numai dintr-un număr finit de astfel de mulțimi, ceea ce demonstrează afirmația.

Corolar. Dacă, în particular, A este finită, atunci există un indice $n_0 \in N$, astfel încît $A \subset F(x_n)$, oricare ar fi $n > n_0$.

Propoziția 9. Dacă multifuncția (E, Y, F) are limită analitică în punctul x_0 și $(a) - \lim_{x \rightarrow x_0} F(x) \subset B$, $B \in \mathcal{P}(Y)$, atunci pentru orice șir $(x_n) \subset E$, $x_n \neq x_0$, $x_n \rightarrow x_0$, are loc $\bigcap_{n > n_0} F(x_n) \subset B$, $n_0 \in N$, fix.

Demonstrație. Fie $(a) - \lim_{x \rightarrow x_0} F(x) = A \subset B$. Orice punct care aparține unei intersecții de tipul celei din enunț, aparține lui A și, implicit, lui B . Din propozițiile 8 și 9, decurge o altă afirmație.

Propoziția 10. Dacă multifuncția (E, Y, F) are limită analitică în punctul x_0 și $A \subset (a) - \lim_{x \rightarrow x_0} F(x) \subset B$, atunci pentru orice $n_0 \in N$, fixat, are loc $\bigcap_{n > n_0} F(x_n) \subset B$, unde $(x_n) \subset E$, $x_n \neq x_0$, $x_n \rightarrow x_0$ și $x \in A$, arbitrar.

În aceeași ordine de idei se poate enunța și demonstra

Propoziția 11. Fie multifuncțiile (E, Y, F) și (E, Y, G) cu limită analitică în punctul x_0 . Dacă $(a) - \lim_{x \rightarrow x_0} F(x) \subset (a) - \lim_{x \rightarrow x_0} G(x)$, atunci oricare ar fi șirul $(x_n) \subset E$, $x_n \neq x_0$, $x_n \rightarrow x_0$ și oricare ar fi numărul natural n_0 , are loc relația $\bigcap_{n > n_0} F(x_n) \subset \bigcap_{n > n_0} G(x_n)$.

Următoarea propoziție apare ca un criteriu de existență a limitei analitice.

Propoziția 12. Fie multifuncția (E, Y, G) cu limită analitică vidă în x_0 . Dacă există o mulțime $A \in \mathcal{P}(Y)$, astfel încît $F(x_n) \Delta A \subseteq G(x_n)$, oricare ar fi $(x_n) \subset E$, $x_n \neq x_0$, $x_n \rightarrow x_0$ și oricare ar fi $n > n_0$, n_0 fix, atunci $(a) - \lim_{x \rightarrow x_0} F(x) = A$.

Demonstrație. Pentru orice $n > n_0$, există un șir de mulțimi $(D_n) \subset \mathcal{P}(Y)$, astfel încît $F(x_n) \Delta A \subseteq G(x_n) \subseteq D_n \downarrow \Phi$. Rezultă că $(a) - \lim_{x \rightarrow x_0} F(x_n) \Delta A = \Phi$, ceea ce este echivalent cu $(a) - \lim_{x \rightarrow x_0} F(x_n) = A$ (v. [5]).

2. Algebra Boole $\mathcal{L}(E, Y, x_0)$. Notăm prin $\mathcal{L}(E, Y, x_0)$ mulțimea tuturor multifuncțiilor definite pe $E \subset X$, X spațiu de convergență, care au limită analitică în punctul de acumulare al mulțimii E , x_0 .

Sînt cunoscute operațiile de reuniune și intersecție a multifuncțiilor. Introducem în plus operația de complementariere a unei multifuncții.

Definiție. Fie multifuncția (E, Y, F) . Prin complementara acesteia se înțelege multifuncția $(E, Y, \bar{F})$, definită prin $\bar{F}(x) = \overline{F(x)}$, oricare ar fi $x \in E$. Deoarece mulțimea $\mathcal{P}(Y)$, $Y \neq \Phi$, este o algebră Boole în raport cu operațiile de reuniune, intersecție și complementariere, familia tuturor multifuncțiilor definite pe E cu valori în Y este, de asemenea, o algebră Boole în raport cu operațiile amintite.

Propoziția 13. Mulțimea $\mathcal{L}(E, Y, x_0)$, înzestrată cu operațiile de reuniune, intersecție și complementariere, este o algebră Boole.

Demonstrație. Este suficient să arătăm că această mulțime este închisă în raport cu cele trei operații. Fie F și G două multifuncții din mulțimea $\mathcal{L}(E, Y, x_0)$. Oricare ar fi șirul $(x_n) \subset E$, $x_n \neq x_0$, $x_n \rightarrow x_0$, avem :

$$1) (a) - \lim_{x \rightarrow x_0} (F \cup G)(x) = (a) - \lim_{x \rightarrow x_0} F(x) \cup G(x) = (a) - \lim_n F(x_n) \cup G(x_n) = \\ = [(a) - \lim_n F(x_n)] \cup [(a) - \lim_n G(x_n)] = [(a) - \lim_{x \rightarrow x_0} F(x)] \cup [(a) - \lim_{x \rightarrow x_0} G(x)];$$

$$2) (a) - \lim_{x \rightarrow x_0} (F \cap G)(x) = (a) - \lim_{x \rightarrow x_0} F(x) \cap G(x) = \\ = (a) - \lim_n F(x_n) \cap G(x_n) = [(a) - \lim_n F(x_n)] \cap [(a) - \lim_n G(x_n)] = \\ = [(a) - \lim_{x \rightarrow x_0} F(x)] \cap [(a) - \lim_{x \rightarrow x_0} G(x)];$$

$$3) (a) - \lim_{x \rightarrow x_0} \overline{F}(x) = (a) - \lim_{x \rightarrow x_0} [Y - F(x)] = (a) - \lim_n [Y - F(x_n)] = \\ = [(a) - \lim_n Y] - [(a) - \lim_n F(x_n)] = Y - [(a) - \lim_{x \rightarrow x_0} F(x)] = \\ = (a) - \lim_{x \rightarrow x_0} F(x).$$

Deci oricare ar fi $F, G \in \mathcal{L}(E, Y, x_0)$ au loc apartenențele $F \cup G \in \mathcal{L}(E, Y, x_0)$, $F \cap G \in \mathcal{L}(E, Y, x_0)$ și $\overline{F} \in \mathcal{L}(E, Y, x_0)$, ceea ce arată că $\mathcal{L}(E, Y, x_0)$ este închisă în raport cu cele trei operații. Prin urmare, ea apare ca o subalgebră Boole a algebrei Boole a tuturor multifuncțiilor definite pe E cu valori Y , notată $(\mathcal{F}(E, Y), \mathcal{L}(E, Y, x_0) \subset \mathcal{F}(E, Y))$.

De aceeași manieră se demonstrează și

Propoziția 14. Dacă $F, G \in \mathcal{L}(E, Y, x_0)$ și $A, B \in \mathcal{P}(Y)$, atunci $F - G, F \Delta G$ și $A \cap F$ aparțin de asemenea algebrei Boole, $\mathcal{L}(E, Y, x_0)$.

Demonstrație. Într-adevăr, pentru orice șir $(x_n) \subset E$, $x_n \neq x_0$, $x_n \rightarrow x_0$, avem :

$$1) (a) - \lim_{x \rightarrow x_0} (F - G)(x) = (a) - \lim_{x \rightarrow x_0} [F(x) - G(x)] = (a) - \lim_n [F(x_n) - G(x_n)] = \\ = [(a) - \lim_n F(x_n)] - [(a) - \lim_n G(x_n)] = [(a) - \lim_{x \rightarrow x_0} F(x)] - [(a) - \lim_{x \rightarrow x_0} G(x)];$$

$$2) (a) - \lim_{x \rightarrow x_0} (F \Delta G)(x) = (a) - \lim_{x \rightarrow x_0} [F(x) \Delta G(x)] = (a) - \lim_n [F(x_n) \Delta G(x_n)] = \\ = [(a) - \lim_n F(x_n)] \Delta [(a) - \lim_n G(x_n)] = [(a) - \lim_{x \rightarrow x_0} F(x)] \Delta [(a) - \lim_{x \rightarrow x_0} G(x)];$$

$$3) (a) - \lim_{x \rightarrow x_0} (A \cap F)(x) = (a) - \lim_{x \rightarrow x_0} A \cap F(x) = (a) - \lim_n A \cap F(x_n) = \\ = A \cap [(a) - \lim_n F(x_n)] = A \cap [(a) - \lim_{x \rightarrow x_0} F(x)].$$

Propoziția 15. Fie (E, Y, G) și (E, X, F) două multifuncții. Presupunem că (E, Y, F) are limită analitică vidă în punctul x_0 . Dacă pentru orice șir $(x_n) \subset E$, $x_n \neq x_0$, $x_n \rightarrow x_0$, există un număr natural n_0 , astfel încât $n > n_0$ atrage $G(x_n) \subseteq A \in \mathcal{P}(Y)$, atunci multifuncția $F \cap G$ are limită analitică vidă în punctul x_0 .

Demonstrație. Pentru orice șir $(x_n) \subset E$, $x_n \neq x_0$, $x_n \rightarrow x_0$, avem $(a) - \lim_{x \rightarrow x_0} (F \cap G)(x) = (a) - \lim_n F \cap G(x_n)$. Pe de altă parte, $(F \cap G)(x_n) = F(x_n) \cap G(x_n) \subseteq A \cap D_n \subseteq D_n \downarrow \Phi$ pentru $n > n_0$. Deci $(a) - \lim_n (F \cap G)(x_n) = \Phi$ și propoziția este demonstrată.

3. Noțiunea de continuitate analitică a multifuncțiilor. Considerăm mulțimea oarecare Y , spațiul de convergență X și $E \subset X$.

Definiția 1. a) Spunem că o multifuncție (E, Y, F) este analitic continuă în punctul $x_0 \in E$ dacă și numai dacă pentru orice șir $(x_n) \subset E$, $x_n \rightarrow x_0$, are loc relația $F(x_n) \xrightarrow{(a)} F(x_0)$.

b) Punctele din E în care F nu este continuă, se numesc puncte de (a) -discontinuitate.

Observații. a) Rezultă imediat echivalența logică: $[(E, Y, F) \text{ este analitic continuă în } x_0 \in E] \Leftrightarrow [(a) - \lim_{x \rightarrow x_0} F(x) = F(x_0)]$.

b) Dacă x_0 este un punct izolat al lui E , atunci orice șir $(x_n) \subset E$, $x_n \rightarrow x_0$ are începînd cu un anumit indice termenii egali cu x_0 . Prin urmare șirul $(F(x_n))$ pentru $n > n_0$, n_0 fix, se reduce la șirul $(F(x_0))$, care este analitic convergent către $F(x_0)$. Deci F este continuă în orice punct izolat al lui E .

Definiția 2. Dacă multifuncția (E, Y, F) este continuă în toate punctele unei mulțimi $A \subset E$, vom spune că ea este continuă pe A .

Definiția 3. Fie multifuncția (E, Y, F) și $x_0 \in E \subset X$, unde X este un spațiu de convergență. Spunem că F este (a) -continuă în punctul $x_0 \in A \subset E$ relativ la mulțimea A , dacă pentru orice șir $(x_n) \subset A$, $x_n \rightarrow x_0$ avem $F(x_n) \xrightarrow{(a)} F(x_0)$.

Observații. a) Condiția din această definiție este echivalentă cu a spune că $(a) - \lim_{\substack{x \rightarrow x_0 \\ x \in A}} F(x) = F(x_0)$.

b) Dacă F este (a) -continuă în x_0 , atunci ea este (a) -continuă în x_0 relativ la orice submulțime $A \subset E$.

c) Dacă $A \subset E$ este de forma $V \cap E$, unde $V \in \mathcal{O}(x_0)$ atunci (a) -continuitatea lui F în x_0 relativ la A , atrage (a) -continuitatea ei în x_0 , deoarece orice șir $(x_n) \subset E$, $x_n \rightarrow x_0$ are toți termenii, cu excepția unui număr finit, cuprinși în $A = V \cap E$.

Definiția 4. Fie $(X, <)$ o mulțime total ordonată și fie (E, Y, F) o multifuncție, unde $E \subset X$.

a) F se numește (a) -continuă la stînga în x_0 dacă pentru orice șir $(x_n) \subset E$, $x_0 < x_n$, $x_n \rightarrow x_0$ are loc $F(x_n) \xrightarrow{(a)} F(x_0)$.

b) F se numește (a) -continuă la dreapta în x_0 , dacă pentru orice șir $(x_n) \subset E$, $x_0 < x_n$, $x_n \rightarrow x_0$ are loc $F(x_n) \xrightarrow{(a)} F(x_0)$.

Propoziția 16. Fie $(X, <)$ o mulțime liniar ordonată și fie multifuncția (E, Y, F) , $E \subset X$. F este (a) -continuă în punctul $x_0 \in E$ dacă și numai dacă ea este (a) -continuă la dreapta și la stînga în acest punct.

Demonstrație. Dacă F este (a) -continuă în x_0 (relativ la E), atunci ea este (a) -continuă în x_0 relativ la orice submulțime $A \subset E$, în particular, la stînga și la dreapta. Reciproc, presupunem că există $\lim_{\substack{x \rightarrow x_0 \\ x < x_0}} F(x) =$

$= F(x_0)$ și $\lim_{\substack{x \rightarrow x_0 \\ x_0 < x}} F(x) = F(x_0)$. Orice șir $(x_n) \subset E$, $x_n \rightarrow x_0$ poate fi reprezentat

ca reuniune de șiruri crescătoare sau descrescătoare la x_0 . Or, pentru aceste două clase disjuncte de șiruri, $F(x_n) \xrightarrow{(a)} F(x_0)$, ceea ce asigură (a) -continuitatea lui F în $x_0 \in E$.

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LA LIMITE ET LA CONTINUITÉ ANALYTIQUE DES MULTI-APPLICATIONS (II)

(Résumé)

On démontre certaines propriétés des multi-applications qui ont une limite analytique dans un point, on définit la notion de „continuité analytique” dans un point et on déduit la structure d'algèbre Boole des familles de multi-applications $\mathcal{L}(E, Y, x_0)$.

ON THE SYMBOLIC REPRESENTATION OF DISCRETE SIGNALS

BY

DANIEL CONDURACHE

1. Let φ_a be a class of functions $f: \mathbb{Z} \rightarrow \mathbb{R}$, also denoted as discrete signals, with the property:

$$(1) \quad f(\tau) \in \varphi_a \Rightarrow f(\tau+1) \in \varphi_a.$$

Let $\mathcal{A}_n$ denote an n order algebra over the real number field $\mathbb{R}$ [2] with the multiplication table

$$(2) \quad \varepsilon_i \varepsilon_j = \sum_{k=1}^n \gamma_{ijk} \varepsilon_k$$

with $\varepsilon_i \in \mathcal{A}_n$, $i = \overline{1, n}$ as elements belonging to a certain basis and $\gamma_{ijk} \in \mathbb{R}$, $i, j, k = \overline{1, n}$ its constants of structure.

Let

$$(3) \quad \hat{w} = \sum_{i=1}^n a_i \varepsilon_i; \quad \hat{w}^* = \sum_{i=1}^n a_i^* \varepsilon_i; \quad a_i, a_i^* \in \mathbb{R}, \quad i = \overline{1, n}$$

be two elements belonging to $\mathcal{A}_n$ which may be established in a correspondence with $f(\tau)$, respectively $f(\tau+1)$ in φ_a .

We may say that the elements (3_1) of algebra $\mathcal{A}_n$ represent symbolically the functions (1_1) if

$$(4) \quad \hat{w}^* = \hat{D} \hat{w},$$

where

$$(5) \quad \hat{D} = \sum_{i=1}^n d_i \varepsilon_i; \quad d_i \in \mathbb{R}, \quad i = \overline{1, n}$$

is an element fixed in the algebra with the role of a *right-shift operator* for the functions (1_1) .

This symbolic representation shall be denoted as $f(\tau) \leftrightarrow \hat{w}$. With (3)

and (5), (4) becomes $\hat{w}^* = \sum_{j=1}^n \sum_{k=1}^n a_j \beta_{jk} \varepsilon_k \left(= \sum_{i=1}^n a_i^* \varepsilon_i \right)$, with

$$(6) \quad \beta_{jk} = \sum_{i=1}^n d_i \gamma_{ijk}; \quad a_k = \sum_{j=1}^n \beta_{jk} a_j \quad (j, k = \overline{1, n}).$$

φ_a being a linear space, as considered from the point of view of the addition of its elements and their multiplication with scalars, the correspondence we impose represents a homomorphism.

The class of the discrete signals representable symbolically by means of this proceeding is a non-empty one.

So, let $f_j \in \varphi_a (j = \overline{1, n})$ be the functions representable symbolically by the elements belonging to the basis of algebra (2): $f_j(\tau) \leftrightarrow \varepsilon_j (j = \overline{1, n})$.

If $f_j(\tau+1) \leftrightarrow \hat{w}_j^* (j = \overline{1, n})$, then

$$(7) \quad \hat{w}_j^* = \hat{D} \varepsilon_j = \sum_{k=1}^n \beta_{jk} \varepsilon_k \quad (j = \overline{1, n}),$$

with $\beta_{jk} (j, k = \overline{1, n})$ given by (6₁). According to the homomorphism in (7), we obtain

$$(8) \quad f_j(\tau+1) = \sum_{k=1}^n \beta_{jk} f_k(\tau) \quad (j = \overline{1, n}).$$

Consequently, the functions $f_j(\tau)$, $j = \overline{1, n}$ verify a homogeneous system of difference equations with constants, 1st order coefficients.

The function (1₁) representable symbolically by element w is:

$$(9) \quad f(\tau) = \sum_{j=1}^n a_j f_j(\tau), \quad a_j \in \mathbb{R} \quad (j = \overline{1, n}).$$

The proposition that follows has thus been demonstrated.

Proposition. *The class φ_a of discrete signals representable symbolically by the elements of an n order finite algebra over the real number field is the class of functions solutions to homogeneous n order difference equations with constant coefficients*

$$(10) \quad B_0 y(n+\tau) + B_1 y(n-1+\tau) + \dots + B_{n-1} y(\tau+1) + B_n y(\tau) = 0,$$

$$B_k \in \mathbb{R} \quad (k = \overline{1, n}).$$

The above equation is equivalent to (8).

2. If algebra (2) is an associative one and has a unity, then an isomorphism with a matrix algebra [2] takes place.

We denote with

$$(11) \quad \hat{w} \leftrightarrow [b_{jk}]_{j,k=\overline{1,n}} \text{ not } [\hat{w}]$$

that the $[b_{jk}]_{j,k=\overline{1,n}}$ n order square matrix corresponds by isomorphism to element $\hat{w} \in \mathcal{A}_n$.

If $\hat{w} = \sum_{i=1}^n a_i \varepsilon_i$; $a_i \in \mathbb{R}$ ($i = \overline{1, n}$), then [2]:

$$(12) \quad b_{jk} = \sum_{i=1}^n a_i \gamma_{ijk} \quad (k, j = \overline{1, n}).$$

It results that to element (5) which has a right-shift role for the functions $f(\tau)$ corresponds by isomorphism the matrix $[\hat{D}] = [\beta_{jk}]_{j,k=\overline{1,n}}$ with $\beta_{jk} (j,k=\overline{1,n})$ given by relation (6₁).

Consequently, the matrix of the coefficients of the difference system (8) is that corresponding by isomorphism to element (5).

For a certain element $\hat{w} \in \mathcal{A}_n$ belonging to an associative algebra with the unity element $\hat{u}$, $\exp \hat{w}$ may be defined as [1]

$$(13) \quad \exp \hat{w} = \sum_{k=0}^{\infty} \frac{\hat{w}^k}{k!}; \quad \hat{w}^0 = \hat{u}.$$

Taking into account the definition of the exponential function for a matrix, the following implication becomes obvious:

$$(14) \quad \hat{w} \leftarrow \div \rightarrow [\hat{w}] \Rightarrow \exp \hat{w} \leftarrow \div \rightarrow \exp[\hat{w}]$$

3. In [1] a symbolic representation of continual signals on finite algebras over the real number field is introduced. This way, a function $f: \mathbb{R} \rightarrow \mathbb{R}$ is representable symbolically on algebra $\mathcal{A}_n$ by the element $\hat{w}$ if to its derivative $\dot{f}(t)$ corresponds the element $\hat{d}\hat{w}$. The element $\hat{d} = \sum_{i=1}^n d_i \varepsilon_i$ is fixed in $\mathcal{A}_n$ and plays the role of a derivation operator for the functions $f(t)$.

The class of functions representable by means of this proceeding is that of functions that are solutions to homogeneous, linear differential equations with constant coefficients.

The two symbolic representations are linked by the following

Theorem. *If $f: \mathbb{R} \rightarrow \mathbb{R}$ is representable symbolically on the associative algebra with unity $\mathcal{A}_n$ by the element $\hat{w} \in \mathcal{A}_n$ with the derivation operator $\hat{d}$, then the restriction $f_\tau: \mathbb{Z} \rightarrow \mathbb{R}$ is representable symbolically by the same element of the algebra with the right-shift operator $\hat{D} = \exp \hat{d}$.*

Proof. If $f(t)$ is representable symbolically on algebra $\mathcal{A}_n$ by the element $\hat{w} = \sum_{i=1}^n a_i \varepsilon_i$ with the derivation operator $\hat{d} = \sum_{i=1}^n d_i \varepsilon_i$, then

$$(15) \quad f(t) = \sum_{j=1}^n a_j f_j(t).$$

The functions $f_j(t)$, $j=\overline{1,n}$ represent a particular solution to the system

$$(16) \quad \dot{f}_j(t) = \sum_{k=1}^n \alpha_{jk} f_k(t), \quad j=\overline{1,n},$$

where $\alpha_{jk} = \sum_{i=1}^n d_i \gamma_{ijk}$; $j,k=\overline{1,n}$ and also $[\alpha_{jk}]_{j,k=\overline{1,n}} \leftarrow \div \rightarrow \hat{d}$.

The solution to system (15) is

$$(17) \quad f_j(t)_{j=\overline{1,n}} = \exp \{ [\alpha_{jk}]_{j,k=\overline{1,n}} t \} C,$$

where C is a column-matrix of n real constants.

The restriction $f_\tau: \mathbb{Z} \rightarrow \mathbb{R}$ of the function (15)

$$(18) \quad f(\tau) = \sum_{j=1}^n a_j f_j(\tau), \quad \tau \in \mathbb{Z}$$

is representable symbolically by element $\hat{w} = \sum_{j=1}^n a_j \varepsilon_j$ with the functions $f_j(\tau)$, $(j=\overline{1,n})$ satisfying the system of difference equations

$$(19) \quad f_j(\tau+1) = \sum_{k=1}^n \beta_{jk} f_k(\tau), \quad \beta_{jk} \in \mathbb{R}, \quad j, k = \overline{1,n}$$

and the right-shift operator $\hat{D}$ corresponding by isomorphism to the matrix of the coefficients of system (19).

The solution to system (19) is

$$(20) \quad f_j(\tau)_{j=\overline{1,n}} = \{ [\beta_{jk}]_{j,k=\overline{1,n}} \}^\tau C,$$

where C is a column-matrix with n real constants.

Making $t \Rightarrow \tau$ in (17), from (17) and (20) it results

$$(21) \quad \exp \{ [\alpha_{jk}]_{j,k=\overline{1,n}} \tau \} = \{ [\beta_{jk}]_{j,k=\overline{1,n}} \}^\tau.$$

For $\tau=1$ (21) becomes

$$[\beta_{jk}]_{j,k=\overline{1,n}} = \exp [\alpha_{jk}]_{j,k=\overline{1,n}}.$$

Taking into account:

$$[\beta_{jk}]_{j,k=\overline{1,n}} \xrightarrow{\hat{D}}, \quad [\alpha_{jk}]_{j,k=\overline{1,n}} \xrightarrow{\hat{d}}$$

and (14) one obtains $\hat{D} = \exp \hat{d}$ q.e.d.

Remarks. 1. If $f: \mathbb{R} \rightarrow \mathbb{R}$ is a continual signal, its restriction $f_\tau: \mathbb{Z} \rightarrow \mathbb{R}$ represents the discrete signal resulted from the sampling [3] of signal $f(t)$. The symbolic representation suggested in this paper is applicable to a class larger than that of signals resulted from the sampling of continual signals representable symbolically in the way of [1].

This statement is justified by the fact that the class of discrete signals resulted from the sampling of the functions solutions to homogeneous n order differential equations with constant coefficients is less large than that of the functions solutions to n order homogeneous difference equations.

2. Because of the homomorphism of the there imposed correspondence, it becomes obvious the following

Proposition 2. If $f(\tau) \leftrightarrow \hat{w}$ with the right-shift operator $\hat{D}$, then

$$\Delta_h f(\tau) = f(\tau+h) - f(\tau) \longleftrightarrow (\hat{D}^h - \hat{u}) \hat{w}.$$

Due to the proposition above, and to the symbolic representation suggested here, solutions to non-homogenous difference equations may be obtained.

4. Conclusions. 1. A symbolic representation of discrete signals on finite algebras over the real number field is suggested.

The paper also offers a theorem allowing the transfer of the most important results concerning continual signals in to the whole class of discrete signals resulted from the sampling of those from [1]. The proceeding is also applicable to a large class of discrete signals: those who represent solutions to homogeneous difference equations.

2. The results are to be applied to problems concerning difference equations, transmitted message, digital signal processing, etc.

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O REPREZENTARE SIMBOLICĂ A SEMNALELOR DISCRETE

(Rezumat)

Se propune o reprezentare simbolică a unor semnale discrete pe algebre finite pe corp real. Clasa semnalelor reprezentabile simbolic prin acest procedeu este aceeași cu clasa funcțiilor soluții de ecuații cu diferențe finite omogene cu coeficienți constanți. Se stabilesc de asemenea rezultate ce pun în corespondență reprezentarea propusă cu reprezentări simbolice cunoscute ale unor semnale continue.

AN EFFICIENT HEURISTIC METHOD FOR THE FINDING OF A MINIMAL HAMMILTONIAN CIRCUIT IN CONNECTION WITH A TRANSPORT PROBLEM

BY

OCTAV BRUDARU

1. The Problem. Moreover the wellknown transport problems, in practice it appears one more special problem described as follows.

Let $C_1, \dots, C_n$ be the centres who request products respectively in the quantities $q_1, \dots, q_n$. These products are delivered from a single warehouse C_0 which can satisfy the demands'sum,

$$(1) \quad Q = \sum_{i=1}^n q_i.$$

For any $i, j \in \{0, 1, \dots, n\}$ the numbers $p_{ij} \geq 0$ and $p_{ij}^0 \geq 0$ represent the cost of the transport for a useful and respectively, residual weight equal to the unit, from C_i to C_j . The provisioning is made with a vehicle of which weight is Q_0 and, which can carry at least the quantity Q , and if it reaches the C_j centre, then it unloads here just the q_j quantity. If $i, j \in \{0, 1, \dots, n\}$ and $i \neq j$ then the cost of the transport for the quantity q (where $0 \leq q \leq Q$) from C_i to C_j is $Q_0 p_{ij}^0 + q p_{ij}$. The total cost is, of course, the sum of intermediate costs.

Let us find an order to supply the centres, with the following conditions:

- (i) the charged vehicle with Q weight leaves from C_0 ;
- (ii) it passes just once through each centre;
- (iii) it satisfies completely all the demands;
- (iv) it returns to C_0 ;
- (v) the total cost is minimal.

2. Mathematical Model. In order to elaborate a mathematical model let us remark that it is possibly a „zero-one“ formulation. We can introduce the matrix $X = (x_{ij})$ with n rows and n columns where if C_j is the i -th provisioned centre then $x_{ij} = 1$ else $x_{ij} = 0$. Since it passes just once through each centre it follows that (if we assume that any order to supply the centres is possible)

$$(2) \quad \sum_{i=1}^n x_{ij} = 1 \text{ for } j = 1, 2, \dots, n \text{ and } \sum_{i=1}^n x_{ij} = 1 \text{ for } i = 1, 2, \dots, n.$$

The total cost is denoted by $f(X)$, where

$$(3) \quad f(X) = Q_0 \sum_{j=1}^n x_{ij} p_{0j}^0 + Q \sum_{j=1}^n x_{1j} p_{0j} + \sum_{k=1}^n \sum_{j=1}^n \sum_{t=1}^n \left[x_{kj} p_{jt}^0 x_{k+1,t} Q_0 + \right. \\ \left. + x_{kj} p_{jt} x_{k+1,t} \left(Q - \sum_{r=1}^n \sum_{s=1}^n x_{rs} q_s \right) \right] + Q_0 \sum_{j=1}^n x_{nj} p_{j0}^0.$$

Therefore, we have obtained a pseudo-boolean programme with a non-linear function for which the efficient computational techniques are not available. For example, in [1] and [4] are described different methods for solving such a programme, but the computation is very laboriously. In practice, a simple and speedily method is very useful, although it offers a suboptimal solution. In this order, let us attach to our problem an oriented graph $G=(X, U)$ as follows:

a) $X = \{0, 1, \dots, n\}$;

b) if there exists a connection from C_i to C_j , where $i, j \in X$, and $i \neq j$ then $(i, j) \in U$.

We define the functions of unitary costs $p: U \rightarrow R$ as $p(i, j) = p_{ij}$, $p^0: U \rightarrow R$ as $p^0(i, j) = p_{ij}^0$ and the function of the demand $q: X - \{0\} \rightarrow R$ as $q(i) = q_i$. If $(i, j) \in U$ and q is an useful weight then the cost of the transport using this edge is

$$(4) \quad k(Q_0, q; i, j) = Q_0 p_{ij}^0 + q p_{ij}.$$

If the sequence $[i_0, i_1, \dots, i_n, i_{n+1}]$ (where $i_0 = i_{n+1}$, and $i_j \neq i_k$ for $j \neq k$), is in accordance with (i) -- (v) then the cost of this path is

$$(5) \quad c(i_1, \dots, i_n) = \sum_{k=0}^n [Q_0 p^0(i_k, i_{k+1}) + p(i_k, i_{k+1}) \sum_{j=k+1}^n q(i_j)]$$

where, for $k=n$

$$(6) \quad \sum_{j=k+1}^n q(i_j) = 0.$$

Obviously, the solving of this problem leads to finding an hamiltonian circuit of the minimal cost.

This transport problem is an extension of the wellknown „travelling salesman problem“, because if $q_i = 0$ for $i=1, 2, \dots, n$, $Q=1$, $p_{ij}=0$ for $i, j=0, 1, \dots, n$ and we interpret p_{jk}^0 as the length of the edge (j, k) , then the cost of the transport of any one path is it's length.

In the sequel we shall present an algorithm in order to obtain a sub-optimal hamiltonian circuit.

3. Algorithm. The detailed steps of the procedure are presented below.

Step 1. Determine the set $X_1 = \{i/(0, j) \in U\}$ and for each $x_1 \in X_1$, compute $G_1(x_1) = k(Q_0, Q; 0, x_1)$, $r_1(x_1) = Q - q(x_1)$ and define $d_1^*(x_1) = 0$;

Step 2. For $i=2, 3, \dots, n$ perform:

2.i.1. For each $x \in X - \{0\}$ perform:

2.i.1.1. Determine the set $A_i(x)$, defined by

$$A_i(x) = \{j/j \in X_{i-1} - \{x\}, (j, x) \in U\};$$

2.i.1.2. Determine the set $B_i(x)$ formed of the vertices $j \in A_i(x)$ for which there exists a path from x to j , $[x, y_1, \dots, y_k, j]$ so that $d_{i-1}^*(j) = y_k$, $d_{i-2}^*(y_k) y_{k-1}, \dots, d_{i-k-1}^*(y_1) = x$;

2.i.1.3. Determine the set $D_i(x) = A_i(x) - B_i(x)$.

2.i.2. Determine the set $X_i = \{x / D_i(x) \neq \emptyset\}$;

2.i.3. For each $x_i \in X_i$ compute:

$$\begin{aligned} G_i(x_i) &= G_{i-1}(d_i^*) + k(Q_0, r_{i-1}(d_i^*); d_i^*, x_i) = \\ &= \min_{\substack{d_i \in D_i \\ (x_i)}} [G_{i-1}(d_i) + k(Q_0, r_{i-1}(d_i); d_i, x_i)] \end{aligned}$$

and define $d_i^*(x_i) = d_i^*$;

2.i.4. Determine the sequence $[x_i, x_{i-1}^*, \dots, x_2^*]$ where $x_{i-1}^* = d_i^*(x_i)$, $x_{i-2}^* = d_{i-1}^*(x_{i-1}^*)$, ..., $x_2^* = d_3^*(x_3^*)$ and $r_i(x_i) = r_{i-1}(x_{i-1}^*) - q(x_i)$;

Step 3. Final selection.

3.1. Take $i = n+1$, $X_i = \{0\}$, $D_i(0) = X_{i-1}$, and perform 2.i.3.;

3.2. With $d_{n+1}^*(0)$ determine $x_n^* = d_{n+1}^*(0)$, $x_{n-1}^* = d_n^*(x_n^*)$, ..., $x_1^* = d_2^*(x_2^*)$ and terminate.

The sought hamiltonian circuit is $[0, x_1^*, x_2^*, \dots, x_n^*, 0]$ while the cost of this is $G_{n+1}(0)$.

Theorem. If we suppose that G is complete then the following statements are true:

A. For each $k \in \{1, 2, \dots, n\}$ we have $X_k \neq \emptyset$;

B. The sequence $[0, x_1^*, \dots, x_n^*, 0]$ is an hamiltonian circuit.

Proof. (A) For $k=1$ we have $D_1(x) = \{0\}$ for any $x \in X - \{0\}$. Let us suppose that $X_k \neq \emptyset$, for $k < n$ and let us prove that $X_{k+1} \neq \emptyset$. From $X_k \neq \emptyset$ it results that there exists $i_0 \in X - \{0\}$ so that $D_k(i_0) \neq \emptyset$. Let $i^* \in \{1, 2, \dots, n\}$ so that $i^* \neq i_0$ and $D_{k+1}(i^*) = \emptyset$. Since G is complete it follows that $i_0 \in A_{k+1}(i^*)$ and because $D_{k+1}(i^*) = \emptyset$ we have that $A_{k+1}(i^*) \subseteq B_{k+1}(i^*)$. Thus for i_0 there exists a path $[y_1, \dots, y_{k+1}, i_0]$ which fulfills two conditions: (a) $d_k^*(i_0) = y_{k-1}$, $d_{k-1}^*(y_{k-1}) = y_{k-2}, \dots, d_2^*(y_2) = y_1$; (b) there exists an index s , $1 \leq s \leq k-1$ so that $y_s = i^*$. Let us observe that the path $[y_1, \dots, y_{k+1}, i_0]$ contains $k-2$ vertices which differ from i_0 and i^* . The number of vertices which differ from i_0 and i^* and which can be in X_{k+1} is equal to $n-2$. Since $k < n$, there exists at least a vertex z which is not in the above-mentioned path. Therefore $i_0 \notin B_{k+1}(z)$ and because $i_0 \in A_{k+1}(z)$ it follows that $z \in X_{k+1}$.

(B) Let us suppose that in the circuit $[0, x_1^*, \dots, x_n^*, 0]$ we have $x_j^* = x_k^*$ with $j < k$. In accordance with the algorithm we have $x_{k-1}^* \notin B_k(x_k^*)$. But $x_{k-1}^* = d_k^*(x_k^*)$, ..., $x_j^* = d_{j+1}^*(x_{j+1}^*)$, and applying the Step 2.k.1.2., it follows that $x_{k-1}^* \in B_k(x_k^*)$. Thus the path $[x_1^*, \dots, x_n^*]$ contains n distinct vertices and therefore the sequence offered by the algorithm is an hamiltonian circuit.

4. A Numerical Example. Let the consumers C_1, C_2, C_3, C_4 be with the demands 3, 7, 10 and 8 respectively. Q_0 is 20, $P = (p_{ij})$ ($i, j = 0, 1, 2, 3, 4$) is the following

$$P = \begin{bmatrix} 0 & 1 & 2 & 2 & 5 \\ 1 & 0 & 4 & 2 & 3 \\ 3 & 3 & 0 & 2 & 1 \\ 2 & 1 & 2 & 0 & 1 \\ 1 & 1 & 3 & 2 & 0 \end{bmatrix}$$

and $P^0 = (p_{ij}^0) = P$.

Step 1. $X_1 = \{1, 2, 3, 4\}$;

$G_1(1) = 48, r_1(1) = 25$; $G_1(2) = 96, r_1(2) = 21$;

$G_1(3) = 96, r_1(3) = 18$; $G_1(4) = 240, r_1(4) = 20$;

Step 2.2. $D_2(1) = \{2, 3, 4\}, D_2(2) = \{1, 3, 4\}, D_2(3) = \{1, 2, 4\},$

$D_2(4) = \{1, 2, 3\}$; $X_2 = \{1, 2, 3, 4\}$;

$G_2(1) = 134, r_2(1) = 15, d_2^*(1) = 3$; $G_2(2) = 172, r_2(2) = 11,$

$d_2^*(2) = 3$;

$G_2(3) = 138, r_2(3) = 15, d_2^*(3) = 1$; $G_2(4) = 134, r_2(4) = 10,$

$d_2^*(4) = 3$;

Step 2.3. $D_3(1) = \{2, 4\}, D_3(2) = \{1, 3, 4\}, D_3(3) = \emptyset, D_3(4) = \{1, 2, 3\}$;

$X_3 = \{1, 2, 4\}$; $G_3(1) = 164, r_3(1) = 7, d_3^*(1) = 4$;

$G_3(2) = 208, r_3(2) = 8, d_3^*(2) = 3$; $G_3(4) = 173, r_3(4) = 7, d_3^*(4) = 3$;

Step 2.4. $D_4(1) = \emptyset, D_4(2) = \{1, 4\}, D_4(3) = \emptyset, D_4(4) = \{2\}$; $X_4 = \{2, 4\},$

$G_4(2) = 254, r_4(2) = 0, d_4^*(2) = 4$; $G_4(4) = 236, r_4(4) = 0,$

$d_4^*(4) = 2$;

Step 3. $X_5 = \{0\}$;

$D_5(0) = \{2, 4\}$; $G_5(0) = 256, d_5^*(0) = 4$.

The sought circuit is $[0, 1, 3, 2, 4, 0]$ and this cost is 256.

5. Conclusions. The above algorithm could be easily converted into a routine and it constitutes an efficient instrument with a view to improve the quality of the decisions for the activities which lead to situations assimilated to the described transport problem.

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CIRCUITE HAMILTONIENE DE COST MINIM ÎN LEGĂTURĂ CU O PROBLEMĂ DE TRANSPORT

(Rezumat)

Se prezintă o problemă de transport ce constă în aprovizionarea a n centre folosind un singur mijloc de transport de masă dată (masă reziduală) care pleacă dintr-un depozit unic încărcat cu suma cererilor, trece exact o dată prin fiecare centru satisfăcându-i complet cererea și în final, revine în punctul de plecare. Costul transportului între două puncte intermediare ale rutei depinde de masa reziduală, masa utilă transportată și de costurile unitare de transport respective. Costul total este suma costurilor intermediare.

Rezolvarea acestei probleme conduce la determinarea unui circuit hamiltonian de cost minim și în acest sens se prezintă un eficient procedeu euristic.

DESPRE NECONVERGENȚA ȘIRULUI DE APROXIMAȚII SUCCESIVE ÎN PROBLEMA LUI DARBOUX PENTRU ECUAȚIA $u_{xyz} = f(x, y, z, u)$

DE

GEORGETA TEODORU

1. **Introducere.** În această Notă se consideră problema lui Darboux

$$(1) \quad \begin{cases} u_{xyz} = f(x, y, z, u), & (x, y, z) \in [0, a] \times [0, b] \times [0, c] = I; \\ u(0, y, z) = \sigma_1(y, z), & (y, z) \in [0, b] \times [0, c] = I_{23}; \\ u(x, 0, z) = \sigma_2(x, z), & (x, z) \in [0, a] \times [0, c] = I_{13}; \\ u(x, y, 0) = \sigma_3(x, y), & (x, y) \in [0, a] \times [0, b] = I_{12}, \end{cases}$$

în care se presupune că au loc relațiile

$$(2) \quad \begin{cases} \sigma_1(0, z) = \sigma_2(0, z), & z \in I_3 = [0, c]; \\ \sigma_1(y, 0) = \sigma_3(0, y), & y \in I_2 = [0, b]; \\ \sigma_2(x, 0) = \sigma_3(x, 0), & x \in I_1 = [0, a]. \end{cases}$$

Mai mulți autori (v. [2], ..., [6], [9]) au tratat existența și unicitatea soluției problemei (1), în condițiile (2), cu diverse ipoteze asupra funcțiilor ce intervin.

Prezentăm aici o situație de neconvergență a șirului de aproximații succesive.

Notăm cu R mulțimea numerelor reale și cu R^m , $m \in N$, spațiul liniar cu norma

$$\|x\| = \max \{|x_1|, |x_2|, \dots, |x_m|\}, \quad \text{unde } x = (x_1, x_2, \dots, x_m).$$

Fie mulțimea $P \subset R^{m+3}$ definită astfel:

$$P = \{(x, y, z, u) \mid (x, y, z) \in I, u \in R^m\}.$$

Considerăm funcții $f: P \rightarrow R^m$ care satisfac următoarele ipoteze:

(C₁) $f(\cdot, \cdot, \cdot, u): I \rightarrow R^m$ este măsurabilă pentru orice $u \in R^m$.

(C₂) $f(x, y, z, \cdot): R^m \rightarrow R^m$ este continuă pentru aproape orice $(x, y, z) \in I$.

(C₃) Există un număr $M > 0$ astfel încât $\|f(x, y, z, u)\| \leq M$ pentru aproape orice $(x, y, z) \in I$ și pentru orice $u \in R^m$.

2. Spațiul metric fundamental și teoremele de bază. Fie $Q_\alpha \subset P$ o mulțime definită prin

$$Q_\alpha = \{(x, y, z, u) \in P \mid \|u\| \leq \alpha\} \text{ pentru orice } \alpha > 0.$$

Analog cu demonstrația din [1] se poate demonstra

Teorema 1. Fie $f: Q_\alpha \rightarrow R^m$ care satisface $(C_1) - (C_3)$, $\alpha > 0$. Atunci există un șir $\{f_n\}_{n \in N}$ de funcții continue $f_n: Q_\alpha \rightarrow R^m$ astfel încât:

- (i) $\|f_n(x, y, z, u)\| \leq M$ pentru $(x, y, z, u) \in Q_\alpha$, $n = 1, 2, \dots$;
- (ii) $\lim_{n \rightarrow \infty} \max_{\|u\| \leq \alpha} \|f_n(x, y, z, u) - f(x, y, z, u)\| = 0$ pentru aproape orice $(x, y, z) \in I$.

Pe o cale similară cu cea din [7] putem obține

Teorema 2. Fie $f: Q_\alpha \rightarrow R^m$ care satisfac $(C_1) - (C_3)$, pentru $\alpha > 0$. Atunci pentru orice $\varepsilon > 0$ există o funcție $f^\varepsilon: Q_\alpha \rightarrow R^m$ cu proprietățile:

- (i) $\|f^\varepsilon(x, y, z, u)\| \leq M$ pentru $(x, y, z, u) \in Q_\alpha$;
- (ii) $\max_{\|u\| \leq \alpha} \|f^\varepsilon(x, y, z, u) - f(x, y, z, u)\| \leq \varepsilon$ pentru aproape orice $(x, y, z, u) \in I$;

(iii) $f^\varepsilon(x, y, z, u)$ are derivate parțiale continue de orice ordin în raport cu $u_1, u_2, \dots, u_m$, unde $u = (u_1, u_2, \dots, u_m)$.

Să notăm cu $F(P)$ mulțimea tuturor funcțiilor $f: P \rightarrow R^m$ care satisfac $(C_1) - (C_3)$. Definim în mulțimea $F(P)$ o relație de echivalență în felul următor: pentru orice $f_1, f_2 \in F(P)$ considerăm $f_1 \sim f_2$, dacă există o mulțime $A \subset I$, de măsură nulă, astfel încât $f_1(x, y, z, u) = f_2(x, y, z, u)$ pentru orice $u \in R^m$ și $(x, y, z) \in I - A$.

Fie $(\mathcal{F}(P) = F(P)/\sim)$, mulțimea cît a claselor de echivalență determinate de relația de echivalență $\sim$. Pentru $f \in F(P)$ dat, notăm cu $\tilde{f}$ elementul lui $(\mathcal{F}(P))$ care conține f , $f \in \tilde{f}$, $\tilde{f}$ fiind clasa de echivalență a lui f .

Definim în $(\mathcal{F}(P))$ o metrică ρ prin expresia

$$\rho(\tilde{f}_1, \tilde{f}_2) = \|\tilde{f}_1 - \tilde{f}_2\|_{\mathcal{F}},$$

unde

$$\|\tilde{f}\|_{\mathcal{F}} = \iiint_I \sup_u \|f(x, y, z, u)\| dx dy dz, \quad (x, y, z, u) \in P,$$

pentru $\tilde{f}, \tilde{f}_1, \tilde{f}_2 \in (\mathcal{F}(P))$ și $f \in \tilde{f}$.

Lema 1. $(F(P), \rho)$ este un spațiu metric complet.

Demonstrație. Fie $\{f_n\}_{n \in N}$ un șir Cauchy din $(\mathcal{F}(P))$, deci $\|\tilde{f}_n - \tilde{f}_m\|_{\mathcal{F}} \rightarrow 0$ pentru $m, n \rightarrow \infty$.

Să demonstrăm că șirul este convergent și că limita șirului aparține mulțimii $(\mathcal{F}(P))$. Fie $f_n \in \tilde{f}_n$ și $f_m \in \tilde{f}_m$. Deoarece $\|\tilde{f}_n - \tilde{f}_m\|_{\mathcal{F}} \leq \varepsilon$ pentru orice $\varepsilon > 0$, rezultă că există un număr $n(\varepsilon) \in N$ astfel încât

$$\iiint_I \sup_u \|f_n(x, y, z, u) - f_m(x, y, z, u)\| dx dy dz \leq \varepsilon, \quad (x, y, z, u) \in P,$$

pentru $m, n \geq n(\varepsilon)$.

Să luăm $\varepsilon = 1/2^{2k}$, $k \in N$. Fie $\{n_k\}_{k \in N}$ un șir crescător de numere naturale, cu $n_{k-1} \geq n(1/2^{2k})$. Atunci

$$\iiint_I \sup_u \|f_{n_k}(x, y, z, u) - f_{n_{k-1}}(x, y, z, u)\| dx dy dz \leq \frac{1}{2^{2k}}, \quad (x, y, z, u) \in P,$$

pentru $k=1, 2, 3, \dots$.

Considerăm mulțimea $A_k \subset I$, definită în felul următor:

$$A_k = \left\{ (x, y, z, u) \in I \mid \sup_u \|f_{n_k}(x, y, z, u) - f_{n_{k-1}}(x, y, z, u)\| > \frac{1}{2^k} \right\},$$

unde $(x, y, z, u) \in P$.

Deoarece $A_k \subset I$, rezultă că

$$\frac{1}{2^{2k}} \geq \iiint_{A_k} \sup_u \|f_{n_k}(x, y, z, u) - f_{n_{k-1}}(x, y, z, u)\| dx dy dz \geq \frac{1}{2^k} \mu(A_k),$$

pentru $(x, y, z, u) \in P$. S-a notat prin $\mu(A_k)$ măsura mulțimii A_k . Din relația precedentă obținem $\mu(A_k) \leq 1/2^k$.

Fie mulțimea

$$A = \bigcap_{i=1}^{\infty} \bigcup_{k=i}^{\infty} A_k.$$

Deoarece

$$\mu(A) \leq \mu\left(\bigcup_{k=i}^{\infty} A_k\right) \leq \sum_{k=i}^{\infty} \mu(A_k) < \sum_{k=i}^{\infty} \frac{1}{2^k} = \frac{1}{2^{i-1}}, \quad \text{pentru } i=1, 2, \dots$$

avem $\mu(A)=0$. Fie $A^* = I - A$ și $A_k^* = I - A_k$. Avem atunci $A^* = \bigcup_{i=1}^{\infty} \bigcap_{k=1}^{\infty} A_k^*$.

Dacă $(x, y, z) \in A^*$ aceasta implică existența unui număr $i \in N$, astfel încât pentru orice $k \geq i$ să fie satisfăcută inegalitatea

$$\sup_u \|f_{n_k}(x, y, z, u) - f_{n_{k-1}}(x, y, z, u)\| \leq \frac{1}{2^k} \quad \text{pentru } (x, y, z, u) \in P.$$

Deci

$$\sum_{k=i}^{\infty} \sup_u \|f_{n_k}(x, y, z, u) - f_{n_{k-1}}(x, y, z, u)\| < \infty \quad \text{pentru } (x, y, z) \in A^*.$$

Atunci seriile

$$f_{n_0}(x, y, z, u) + \sum_{k=1}^{\infty} [f_{n_k}(x, y, z, u) - f_{n_{k-1}}(x, y, z, u)]$$

sînt absolut și uniform convergente pe A^* independent de $u \in R^m$.

Fie $f: P \rightarrow R^m$ definită prin

$$f(x, y, z, u) = \begin{cases} \lim_{k \rightarrow \infty} f_{n_k}(x, y, z, u) & \text{pentru } (x, y, z) \in A^* \text{ și } u \in R^m \\ 0 & \text{pentru } (x, y, z) \in A \text{ și } u \in R^m. \end{cases}$$

Funcția f satisface condițiile $(C_1)-(C_3)$. Vom demonstra că $\|f_n - f\|_{\mathcal{F}} \rightarrow 0$ pentru $n \rightarrow \infty$. Pentru $n, k > n(\varepsilon)$, considerînd $m = n_k$ și ținînd cont că șirul $\{f_n\}_{n \in \mathbb{N}}$ este șir Cauchy, avem

$$\|f_n - f_{n_k}\|_{\mathcal{F}} = \iiint_I \sup_u \|f_n(x, y, z, u) - f_{n_k}(x, y, z, u)\| \, dx \, dy \, dz \leq \varepsilon$$

pentru $(x, y, z, u) \in P$.

Notînd, pentru n fixat,

$$\psi_k(x, y, z) = \sup_u \|f_n(x, y, z, u) - f_{n_k}(x, y, z, u)\|, \quad \text{cu } (x, y, z, u) \in P,$$

rezultă că

$$\begin{aligned} \|\tilde{f}_n - \tilde{f}\|_{\mathcal{F}} &= \iiint_I \sup_u \|f_n(x, y, z, u) - f(x, y, z, u)\| \, dx \, dy \, dz = \\ &= \iiint_I \sup_u \|f_n(x, y, z, u) - \lim_{k \rightarrow \infty} f_{n_k}(x, y, z, u)\| \, dx \, dy \, dz = \\ &= \iiint_I \lim_{k \rightarrow \infty} \psi_k(x, y, z) \, dx \, dy \, dz. \end{aligned}$$

În virtutea lemei lui Fatou, obținem

$$\begin{aligned} \iiint_I \lim_{k \rightarrow \infty} \psi_k(x, y, z) \, dx \, dy \, dz &\leq \lim_{k \rightarrow \infty} \iiint_I \psi_k(x, y, z) \, dx \, dy \, dz = \\ &= \lim_{k \rightarrow \infty} \|\tilde{f}_n - \tilde{f}_{n_k}\|_{\mathcal{F}} \leq \varepsilon, \quad \text{pentru } n \geq n(\varepsilon). \end{aligned}$$

Cu aceasta demonstrația este completă.

Fie acum $\sigma_1 \in AC(I_{23}, R^m)$, $\sigma_2 \in AC(I_{13}, R^m)$, $\sigma_3 \in AC(I_{12}, R^m)$, unde $AC(X, R^m)$ semnifică mulțimea funcțiilor absolut continue $\sigma: X \rightarrow R^m$. Pe o cale similară cu cea utilizată de A. Alexiewicz [1] și M. Kisielewicz [7] se poate demonstra că pentru orice $\tilde{f} \in \mathcal{F}(P)$ există cel puțin o soluție a problemei (1), în ipoteza că relațiile (2) sînt satisfăcute.

Pentru $\tilde{f} \in \mathcal{F}(P)$ și pentru $\sigma_1, \sigma_2, \sigma_3$ funcții fixate, îndeplinind ipoteza precedentă, considerăm șirul $\{u_n^f\}_{n \in \mathbb{N}}$ de aproximații succesive definit prin relațiile

$$(3) \quad u_{n+1}^f(x, y, z) = \varphi_0(x, y, z) + \int_0^x \int_0^y \int_0^z f(\xi, \eta, \zeta, u_n^f(\xi, \eta, \zeta)) \, d\xi \, d\eta \, d\zeta,$$

$$n = 0, 1, 2, \dots,$$

unde $u_0^f \in C(I)$ și

$$\varphi_0(x, y, z) = \sigma_1(y, z) + \sigma_2(x, z) + \sigma_3(x, y) - \sigma_3(x, 0) - \sigma_3(0, y) - \sigma_1(0, z) + \sigma_1(0, 0).$$

Spunem că șirul $\{u_n^f\}_{n \in \mathbb{N}}$ converge în I , dacă $\{u_n^f(x, y, z)\}_{n \in \mathbb{N}}$ converge pentru orice $(x, y, z) \in I$. Se verifică ușor următoarea afirmație: dacă $f \in \mathcal{F}(P)$ satisface o condiție Lipschitz în raport cu u , atunci $\{u_n^f\}_{n \in \mathbb{N}}$ este uniform convergent către soluția unică a lui (1). Mai rezultă că

$$\max_I \|u_n^f(x, y, z)\| \leq \alpha, \text{ pentru orice } n=0, 1, 2, \dots, \text{ unde}$$

$$\alpha = \max \left\{ \sup_I \|u_0^f(x, y, z)\|, \sup_I \|\varphi_0(x, y, z)\| + abc M \right\}.$$

3. Neconvergența aproximațiilor succesive. Vom demonstra că mulțimea $\mathcal{A} \subset \mathcal{F}(P)$ a acelor funcții f pentru care $\{u_n^f\}_{n \in \mathbb{N}}$ nu este convergent este o mulțime Baire de categoria întâia în spațiul $(\mathcal{F}(P), \rho)$.

Pentru $\tilde{f} \in \mathcal{F}(P)$ și $(x, y, z) \in I$ definim $\Delta(\tilde{f}, x, y, z)$ prin relația

$$\Delta(\tilde{f}, x, y, z) = \limsup_{n \rightarrow \infty} \{\text{diam } E[u_n^{\tilde{f}}(x, y, z)]\},$$

unde

$$E[u_n^{\tilde{f}}(x, y, z)] = \{u_n^{\tilde{f}}(x, y, z), u_{n+1}^{\tilde{f}}(x, y, z), \dots\}.$$

Prin $\text{diam } A$ s-a notat diametrul mulțimii $A \subset R^m$. Este evident că egalitatea $\Delta(\tilde{f}, x, y, z) = 0$ pentru fiecare $(x, y, z) \in I$ este echivalentă cu convergența șirului $\{u_n^{\tilde{f}}\}_{n \in \mathbb{N}}$. Astfel șirul $\{u_n^{\tilde{f}}\}_{n \in \mathbb{N}}$ nu este convergent în I dacă și numai dacă există $(\tilde{x}, \tilde{y}, \tilde{z})$ cu proprietatea $\Delta(\tilde{f}, \tilde{x}, \tilde{y}, \tilde{z}) > 0$.

Fie $\{(x_r, y_r, z_r)\}_{r \in \mathbb{N}}$ un șir de puncte din I , dens în I , și fie mulțimea $\Omega_{M, p, r} \subset \mathcal{F}(P)$ definită astfel:

$$\Omega_{M, p, r} = \left\{ \tilde{f} \in \mathcal{F}(P) \mid \|\tilde{f}\| \leq M, \Delta(\tilde{f}, x_r, y_r, z_r) \geq \frac{1}{p} \right\}.$$

L e m a 2. *Mulțimea $\Omega_{M, p, r}$ este o mulțime închisă a mulțimii $\mathcal{F}(P)$ pentru orice $M, p, r=1, 2, \dots$.*

Demonstratie. Presupunem dat un șir $\{f_k\}_{k \in \mathbb{N}}$ de elemente din $\Omega_{M, p, r}$ astfel încât $\|f_k - \tilde{f}\|_{\mathcal{F}} \rightarrow 0$ pentru $k \rightarrow \infty$, unde $\tilde{f} \in \mathcal{F}(P)$. Din inegalitatea $\|\tilde{f}\| \leq \|\tilde{f} - f_k\| + \|f_k\|$ rezultă imediat că $\|\tilde{f}\|_{\mathcal{F}} \leq M$.

Pentru demonstrarea lemei, trebuie arătat că $\tilde{f} \in \Omega_{M, p, r}$. Există un subșir $\{f_{n_k}\}_{k \in \mathbb{N}}$ al șirului $\{f_n\}_{n \in \mathbb{N}}$ astfel încât $\sup \|f_{n_k}(x, y, z, u) - f(x, y, z, u)\| \rightarrow 0$ când $k \rightarrow \infty$, pentru aproape orice $(x, y, z) \in I$, $(x, y, z, u) \in P$.

Pentru fiecare $k=1, 2, 3, \dots$, avem $\Delta(f_{n_k}, x_r, y_r, z_r) \geq 1/p$. Atunci

$$\sup \left\{ \text{diam } E \left[u_{n+v}^{f_{n_k}}(x_r, y_r, z_r) \right] \right\} \geq \frac{1}{p} \text{ pentru } n, k=1, 2, \dots, v=1, 2, \dots$$

Deci, pentru orice $l=1, 2, 3, \dots$, există v_l astfel încât

$$\text{diam } E \left[u_{n+v_l}^{f_{n_k}}(x_r, y_r, z_r) \right] \geq \frac{1}{p} - \frac{1}{l},$$

adică

$$\sup_{(\alpha, \beta)} \left\| u_{n+v_l+\alpha}^{f_{nk}}(x_r, y_r, z_r) - u_{n+v_l+\beta}^{f_{nk}}(x_r, y_r, z_r) \right\| \geq \frac{1}{p} - \frac{1}{l}.$$

Atunci, pentru fiecare $q=1, 2, \dots$, există (α_q, β_q) astfel încît

$$\left\| u_{n+v_l+\alpha_q}^{f_{nk}}(x_r, y_r, z_r) - u_{n+v_l+\beta_q}^{f_{nk}}(x_r, y_r, z_r) \right\| \geq \frac{1}{p} - \frac{1}{l} - \frac{1}{q}, \text{ pentru } n, k=1, 2, \dots$$

Fie

$$U(k, n+v_l+\alpha_q)(x, y, z) = u_{n+v_l+\alpha_q}^{f_{nk}}(x, y, z).$$

Este ușor de văzut că familia $X \subset C(I, R^m)$ definită prin $X = \{U(k, n+v_l+\alpha_q)\}$, $k, n, l, q=1, 2, \dots$ satisface ipotezele teoremei lui Arzela. Există deci un subșir $\{U(n_k, n+v_l+\alpha_q)\}$ al șirului $\{U(k, n+v_l+\alpha_q)\}$ care este uniform convergent pe I .

Să presupunem că $\lim_{k \rightarrow \infty} U(n_k, n+v_l+\alpha_q) = U(n+v_l+\alpha_q)$ pentru fiecare n, v_l și α_q fixați. Pentru $n, l, q=1, 2, \dots$ și $(x, y, z) \in I$ avem

$$U(n+v_l+\alpha_q)(x, y, z) - \varphi_0(x, y, z) - \iiint_{000}^{xyz} f(\xi, \eta, \zeta, U(n+v_l+\alpha_q-1)(\xi, \eta, \zeta)) d\xi d\eta d\zeta = \sum_{i=1}^3 \Lambda_i(x, y, z),$$

unde am notat

$$\Lambda_1(x, y, z) = U(n+v_l+\alpha_q)(x, y, z) - U(n_k, n+v_l+\alpha_q)(x, y, z),$$

$$\Lambda_2(x, y, z) = \iiint_{000}^{xyz} (f_{n_k}(\xi, \eta, \zeta, U(n_k, n+v_l+\alpha_q-1)(\xi, \eta, \zeta)) - f_{n_k}(\xi, \eta, \zeta, U(n+v_l+\alpha_q-1)(\xi, \eta, \zeta))) d\xi d\eta d\zeta,$$

$$\Lambda_3(x, y, z) = \iiint_{000}^{xyz} [f_{n_k}(\xi, \eta, \zeta, U(n+v_l+\alpha_q-1)(\xi, \eta, \zeta)) - f(\xi, \eta, \zeta, U(n+v_l+\alpha_q-1)(\xi, \eta, \zeta))] d\xi d\eta d\zeta.$$

Deci

$$U(n+v_l+\alpha_q)(x, y, z) = \varphi_0(x, y, z) + \iiint_{000}^{xyz} f(\xi, \eta, \zeta, U(n+v_l+\alpha_q-1)(\xi, \eta, \zeta)) d\xi d\eta d\zeta$$

pentru $(x, y, z) \in I$ și $n, l, q=1, 2, \dots$.

Avem $U(n + v_l + \alpha_q) = u_{n+v_l+\alpha_q}^f$ pentru orice $n, l, q = 1, 2, \dots$. Pentru că

$$\|U(n_k, n + v_l + \alpha_q)(x_r, y_r, z_r) - U(n_k, n + v_l + \beta_q)(x_r, y_r, z_r)\| \geq \frac{1}{p} - \frac{1}{l} - \frac{1}{q}$$

rezultă

$$\|u_{n+v_l+\alpha_q}^f(x_r, y_r, z_r) - u_{n+v_l+\beta_q}^f(x_r, y_r, z_r)\| \geq \frac{1}{p} - \frac{1}{l} - \frac{1}{q}, \text{ pentru orice } n, l, q = 1, 2, \dots$$

Nu este dificil de observat că $\Delta(\tilde{f}, x_r, y_r, z_r) \geq 1/p$. Deci $\tilde{f} \in \Omega_{Mpr}$, adică Ω_{Mpr} este o mulțime închisă.

Lema 3. *Mulțimea Ω_{Mpr} este nedensă în $(\mathcal{F}(P))$ pentru toate valorile $M, p, r = 1, 2, \dots$.*

Demonstrație. Trebuie arătat că $\text{Int } \bar{\Omega}_{Mpr} = \text{Int } \Omega_{Mpr} = \emptyset$; presupunem că există (M, p, r) și o sferă $S_h(\tilde{f}_0) \subset (\mathcal{F}(P))$ cu centrul $\tilde{f}_0 \in (\mathcal{F}(P))$ și raza $h > 0$ astfel încît $S_h(\tilde{f}_0) \subset \bar{\Omega}_{Mpr}$, adică $\tilde{f}_0$ este punct interior pentru $\bar{\Omega}_{Mpr}$. Conform lemei 2, Ω_{Mpr} este o mulțime închisă, deci $\Omega_{Mpr} = \bar{\Omega}_{Mpr}$. Aceasta înseamnă că presupunem $S_h(\tilde{f}_0) \subset \Omega_{Mpr}$. Se observă că pentru orice $\tilde{f} \in \Omega_{Mpr}$ există un număr $\alpha > 0$ astfel încît șirul $\{u_n^f\}_{n \in \mathbb{N}}$ corespunzător lui $\tilde{f}$ este același cu șirul $\{u_n^{\tilde{f}/Q_\alpha}\}_{n \in \mathbb{N}}$ corespunzător lui $\tilde{f}/Q_\alpha$, unde $\tilde{f}/Q_\alpha$ înseamnă restricția funcției $\tilde{f}$ la mulțimea $Q_\alpha = \{(x, y, z, u) \in P \mid \|u\| \leq \alpha\}$.

În virtutea teoremei 2 pentru $\tilde{f}_0/Q_\alpha$ și $\delta > 0$ există o funcție $f^\delta: Q_\alpha \rightarrow R^m$, astfel încît condițiile (i)–(iii) din teorema 2 sînt satisfăcute. Atunci $\max \|f^\delta(x, y, z, u) - f_0(x, y, z, u)\| < \delta$ pentru aproape orice $(x, y, z) \in I$, cu $(x, y, z, u) \in Q_\alpha$.

Luînd $\delta < h/abc$ obținem $\|\tilde{f}^\delta - \tilde{f}_0\| < h$. Rezultă că $\tilde{f}^\delta \in S_h(\tilde{f}_0) \subset \Omega_{Mpr}$. Dar $\tilde{f}$ este lipschitziană în raport cu u , ceea ce implică $\sup_I \Delta(\tilde{f}^\delta, x, y, z) = 0$. Aceasta arată că $\tilde{f}^\delta \notin \Omega_{Mpr}$, deci $\text{Int } \Omega_{Mpr} = \emptyset$.

Teorema 3. *Mulțimea $\mathcal{A}$ a funcțiilor $\tilde{f} \in (\mathcal{F}(P))$ pentru care aproximațiile succesive $\{u_n^f\}_{n \in \mathbb{N}}$ nu converg, este o mulțime Baire de categoria întâia în spațiul metric $(\mathcal{F}(P), \rho)$.*

Demonstrație. În virtutea lemei 3 este suficient de arătat că mulțimea $\mathcal{A}$ se poate reprezenta sub forma

$$\mathcal{A} = \bigcup_{M=1}^{\infty} \bigcup_{p=1}^{\infty} \bigcup_{r=1}^{\infty} \Omega_{Mpr}.$$

Se observă ușor că $\Omega_{Mpr} \subset \mathcal{A}$ pentru orice $M, p, r = 1, 2, \dots$ deci și $\bigcup_{M=1}^{\infty} \bigcup_{p=1}^{\infty} \bigcup_{r=1}^{\infty} \Omega_{Mpr} \subset \mathcal{A}$. Mai rămîne de demonstrat că are loc și incluziunea inversă. Din definiția mulțimii $\mathcal{A}$ rezultă că

$$\mathcal{A} = \{\tilde{f} \in F(P), (\tilde{x}, \tilde{y}, \tilde{z}) \in I \mid \Delta(\tilde{f}, \tilde{x}, \tilde{y}, \tilde{z}) > 0\}.$$

Fie $\tilde{f} \in \mathcal{A}$. Există atunci un întreg pozitiv $\bar{p}$ astfel încît $\Delta(\tilde{f}, \tilde{x}, \tilde{y}, \tilde{z}) \geq 2/\bar{p}$. Pentru un $\bar{p}$ dat, există un element $(\bar{x}, \bar{y}, \bar{z})$ din șirul $\{(x_r, y_r, z_r)\}_{r \in \mathbb{N}}$.

astfel încît $\Delta(\tilde{f}, \bar{x}, \bar{y}, \bar{z}) \geq \Delta(\tilde{f}, \bar{x}, \bar{y}, \bar{z}) - 1/\bar{p}$. Deci, există un întreg pozitiv $\bar{r}$, cu proprietatea $\Delta(\tilde{f}, x_{\bar{r}}, y_{\bar{r}}, z_{\bar{r}}) \geq 1/\bar{p}$.

Este evident că se poate determina un întreg pozitiv $\bar{M}$ astfel încît $\|\tilde{f}_{\mathcal{F}}\| \leq \bar{M}$. Aceasta arată că $\tilde{f} \in \Omega_{\bar{M}\bar{p}\bar{r}}$. Rezultă deci $\mathcal{A} \subset \bigcup_{M=1}^{\infty} \bigcup_{p=1}^{\infty} \bigcup_{r=1}^{\infty} \Omega_{Mpr}$ și demonstrația este încheiată.

Observație. Teorema 3 și teorema lui Baire — orice spațiu metric complet este o mulțime de categoria a doua — implică faptul că mulțimea $\mathcal{B}$ a acelor funcții $\tilde{f} \in (\mathcal{F}(P))$ pentru care șirul aproximațiilor succesive $\{u_n^f\}_{n \in \mathbb{N}}$ este convergent, este o mulțime densă și de categoria a doua în spațiul metric $(\mathcal{F}(P), \rho)$.

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SUR LA NONCONVERGENCE DE LA SUITE DES APPROXIMATIONS
SUCCESSIVES DANS LE PROBLÈME DE DARBOUX POUR L'ÉQUATION $u_{xyz} = f(x, y, z, u)$
(Résumé)

Dans cette Note on démontre que l'ensemble des fonctions $\tilde{f} \in \mathcal{F}(P)$ pour lesquelles la suite des approximations successives $\{u_n^f\}_{n \in \mathbb{N}}$ est convergente à la solution du problème de Darboux (1) est un ensemble dense de deuxième catégorie dans l'espace métrique $(\mathcal{F}(P), \rho)$.

ÉQUATIONS DIFFÉRENTIELLES DE FUCHS, DU DEUXIÈME ORDRE, DANS DES ESPACES DE BANACH

PAR

I. ONCIULESCU

0. Introduction. Dans [3] nous avons étudié une équation différentielle de Fuchs du deuxième ordre avec des fonctions de la variable $z \in \mathcal{O}$ (le corps complexe) à valeurs dans un espace de Banach. Dans cette Note nous considérons un cas plus général, dont la variable se trouve dans un espace linéaire normé arbitraire et les singularités s'introduisent par une fonctionnelle linéaire continue sur l'espace de la variable. La dérivée utilisée dans ces équations est la dérivée de Fréchet.

1. Équations différentielles de Fuchs dans des espaces de Banach avec les dérivées suivant des directions fixées. *Le calcul des intégrales formelles.* On considère un espace linéaire normé X et un espace de Banach Y , sur le corps complexe $\mathcal{O}$.

Soit $h(x)$ une fonctionnelle linéaire continue, définie sur l'espace X , et les fonctions $a(x, u)$, $b(x, u)$ définies sur un domaine $B = \{(x, u), x \in X, u \in Y, \|x\|, \|y\| < a\}$ de $X \times Y$, à valeurs dans Y , par :

$$(1) \quad \begin{cases} a(x, u) = h(x) \sum_{k=0, \infty} h_k(x) a_k(u), \\ b(x, u) = \sum_{k=0, \infty} h^k(x) b_k(u), \end{cases}$$

où a_k, b_k ($k = 0, 1, 2, \dots$) sont des opérateurs linéaires continus définis sur Y à valeurs dans Y et

$$(2) \quad \sum_{k=0, \infty} \|a_k\| \lambda^k < \infty, \quad \sum_{k=0, \infty} \|b_k\| \lambda^k < \infty$$

pour $\lambda \in \mathcal{O}, |\lambda| < a$. Soit aussi $x_0, y_0, z_0 \in X$, tels que $h(x_0), h(y_0), h(z_0) \neq 0$. Alors nous allons montrer que l'équation

$$(3) \quad h^2(x) \frac{d^2 y(x; x_0, y_0)}{dx^2} + a \left(x, \frac{dy(x; x_0)}{dx} \right) + b(x, y) = \Theta$$

sous certaines conditions, analogues aux celles du cas scalaire admet des solutions de la forme

$$(4) \quad y(x) = h^r(x) \sum_{k=0, \infty} h^k(x) c_k,$$

où $r \in \mathcal{C}$ et $c_k \in Y$, $k = 0, 1, 2, \dots$, $c_0 \neq \Theta$.

Nous allons trouver d'abord pour l'équation (3) une solution formelle ayant la forme (4).

En dérivant au sens de Fréchet formellement $y(x)$ on obtient

$$(5) \quad \begin{cases} \frac{dy(x; x_0)}{dx} = \sum_{k=0, \infty} (k+r) h^{k+r-1}(x) h(x_0) c_k, \\ \frac{d^2 y(x; x_0, y_0)}{dx^2} = \sum_{k=0, \infty} (k+r)(k+r-1) h^{k+r-2}(x) h(x_0) h(y_0) c_k. \end{cases}$$

En remplaçant (5) dans (3), compte tenu de (1), on trouve pour $c_0, c_1, \dots, c_n, \dots$ le système d'équations

$$(6) \quad (r+n)(r+n-1)h(x_0)h(y_0)c_n + h(x_0)(r+n)a_n(c_n) + b_n(c_n) + \sum_{m=0, n-1} (h(x_0)(r+m)a_{n-m}(c_m) + b_{n-m}(c_m)) = \Theta.$$

Pour $n=0$, (6) conduit à „l'équation déterminante“

$$(7) \quad r(r-1)h(x_0)h(y_0)c_0 + h(x_0)ra_0(c_0) + b_0(c_0) = \Theta.$$

Supposons que

(I₁) Il existe $r_0 \in \mathcal{C}$, tel que l'égalité (7) soit satisfaite pour tout $c_0 \in Y$ et $2r_0 + n - 1 \neq 0$ ($n = 1, 2, \dots$).

Remarque 1. Si $a_0(u) = \alpha_0 u$, $b_0(u) = \beta_0 u$ où $\alpha_0 \beta_0 \in \mathcal{C}$ et si l'on choisissent $x_0, y_0 \in X$ tels que $h(x_0) = h(y_0) = 1$, alors (7) prend la forme

$$(7') \quad [r(r-1) + \alpha_0 r + \beta_0]c_0 = \Theta.$$

La condition (I₁) est remplie, dans ce cas, si et seulement si r_0 est une racine de l'équation

$$(7'') \quad r(r-1) + \alpha_0 r + \beta_0 = 0.$$

On retrouve de cette manière l'équation déterminante qui apparaît dans le cas scalaire [1].

Soit un $c_0 \in Y$ fixé. Alors les relations (6) ou l'on remplace $r = r_0$, en tenant compte de (I₁) s'écrivent sous la forme

$$(8) \quad \begin{aligned} & n(2r_0 + n - 1)h(x_0)h(y_0)c_n + nh(x_0)a_n(c_n) + \\ & + \sum_{m=0, n-1} (h(x_0)(r+m)a_{n-m}(c_m) + b_{n-m}(c_m)) = \Theta, \end{aligned}$$

d'où

$$(9) \quad c_n = - \frac{h(x_0)a_0(c_n)}{h(x_0)h(y_0)(2r_0 + n - 1)} - \frac{1}{n(2r_0 + n - 1)h(x_0)h(y_0)} \sum_{m=0, n-1} (h(x_0)(r_0 + m)a_{n-m}(c_m) + b_{n-m}(c_m)).$$

On note

$$(10) \quad B_{n-1} = \frac{1}{n(2r_0 + n - 1)h(x_0)h(y_0)} \sum_{m=0, n-1} (h(x_0)(r_0 + m)a_{n-m}(c_m) + b_{n-m}(c_m))$$

et

$$(11) \quad T_n u = - \frac{a_0(u)}{h(y_0)(2r_0 + n - 1)}, \quad n = 1, 2, \dots$$

T_n est un opérateur linéaire défini sur Y à valeurs dans Y . Nous allons supposer que

$$(I_2) \quad \frac{\|a_0\|}{|h(y_0)|\|2r_0 + n - 1|} \leq \alpha < 1 \quad \text{pour } n = 1, 2, \dots$$

Alors (9) s'écrit sous la forme

$$(12) \quad c_n = T_n c_n + B_{n-1}, \quad n = 1, 2, \dots$$

où

$$(13) \quad \|T_n\| \leq \alpha < 1.$$

Par les relations (12) on trouve par récurrence $c_1, c_2, \dots$. De cette façon, sous les hypothèses (I_1) , (I_2) , l'existence des solutions ayant la forme (4) est démontrée.

Remarque 2. Dans le cas particulier $a_0(u) = \alpha_0 u$, $b_0(u) = \beta_0 u$, $h(x_0) = h(y_0) = 1$, l'hypothèse (I_2) peut être remplacée par la condition

$$(I'_2) \quad (2r_0 + n - 1 + \alpha_0) \neq 0, \quad n = 1, 2, \dots$$

Dans ce cas les c_n ($n = 1, 2, \dots$) se déterminent par récurrence dans la manière suivante

$$(12') \quad c_n = \frac{1}{n(2r_0 + n - 1 + \alpha_0)} \beta_{n-1}, \quad n = 1, 2, \dots$$

2. L'existence des solutions de la forme (4). Parce que on a (2), d'après un théorème de Cauchy sur des fonctions analytiques, il résulte qu'il existent M , N et ρ tels que

$$(13) \quad \|a_n\| \leq \frac{M}{\rho^n}, \quad \|b_n\| \leq \frac{N}{\rho^n}.$$

Alors, compte tenu de (12), (13) et (10), on obtient

$$\|c_n\| \leq \frac{1}{1 - \alpha} \|B_{n-1}\| \leq$$

$$\begin{aligned}
 (14) \quad & \leq \frac{1}{1-\alpha} \frac{n}{|2r_0+n-1| \|h(x_0)\| \|h(y_0)\|} \frac{1}{n^2} \sum_{m=0, n-1} \frac{M_1(|r_0|+m)+N}{\rho^{n-m}} \|c_m\| \leq \\
 & \leq \frac{K}{n} \sum_{m=0, n-1} \frac{M_1(|r_0|+m)+N}{n \rho^{n-m}} \|c_m\|,
 \end{aligned}$$

où K est une constante,

$$K \geq \frac{n}{[(1-\alpha)|2r_0+n-1| \|h(x_0)\| \|h(y_0)\|]}, \quad n=1, 2, \dots$$

et $M_1 = M \|h(x_0)\|$. Parce que

$$\frac{M_1(|r_0|+m)+N}{n} \leq \frac{M_1|r_0|+N}{n} + \frac{M_1 m}{n} \leq M_1(|r_0|+1)+N$$

si on note $H = K \max(1, M_1(|r_0|+1)+N)$ on a

$$(15) \quad \|c_n\| \leq \frac{H}{n} \sum_{m=0, n-1} \frac{\|c_m\|}{\rho^{n-m}},$$

d'où on déduit que [1]

$$(16) \quad \|c_n\| \leq \left(\frac{K}{\rho}\right)^n \|c_0\|, \quad n=1, 2, \dots$$

De cette manière il résulte que la série $\sum_{n=0, \infty} \|c_n\| \lambda^n$ est convergente pour

$$|\lambda| < \varepsilon_0.$$

Nous avons donc démontré le

Théorème 1. Les fonctions $a(x, u)$ et $b(x, u)$ étant celles introduites dans le §1, si les conditions (2), (I_1) , (I_2) sont remplies, alors l'équation (3) possède des solutions de la forme (4) où $\sum_{n=0, \infty} \|c_n\| \lambda^n < \infty$ pour $|\lambda| < \varepsilon_0$.

Dans le cas particulier $a_0(u) = \alpha_0 u$, $b_0(u) = \beta_0 u$ (Remarque 1 et 2) la convergence des solutions (4), où c_n se déterminent par (12') se prouve comme dans le cas scalaire [1]. De cette façon il résulte le

Théorème 2. Si

$$(2') \quad \begin{cases} a(x, u) = h(x) \left[\alpha_0 u + \sum_{k=1, \infty} h^k(x) a_k(u) \right], \\ b(x, u) = \beta_0 u + \sum_{k=1, \infty} h^k(x) b_k(u) \end{cases} \quad \text{où } \alpha_0, \beta_0 \in \mathcal{O}$$

et

$$(3') \quad |\alpha_0| + \sum_{k=1, \infty} \|a_k\| \lambda^k < \infty, \quad |\beta_0| + \sum_{k=1, \infty} \|b_k\| \lambda^k < \infty$$

pour $\lambda \in \mathcal{O}$, $|\lambda| < a$ et si $h(x_0) = h(y_0) = 1$ et $r_0(r_0-1) + \alpha_0 r_0 + \beta_0 = 0$ tel que $2r_0 + n - 1 \neq 0$ ($n=1, 2, \dots$) alors l'équation

$$(3) \quad h^2(x)y''(x; x_0, y_0) + a(x, y'(x; x_0)) + b(x, y) = \Theta$$

admet des solutions ayant la forme $y(x) = h^r(x)[c_0 + h_1(x)c_1 + \dots]$ où la série $\sum_{k=0, \infty} \|c_k\| \lambda^k$ est convergente pour $|\lambda| < \varepsilon_0$.

3. Équations de Fuchs dans des espaces de Banach avec les dérivées suivant des directions arbitraires. Des résultats analogues on peut obtenir pour des équations ayant la forme

$$(3') \quad h^2(x) \frac{d^2 y(x; \Delta_1 x, \Delta_2 x)}{dx^2} + a\left(x, \frac{dy(x; \Delta_1 x)}{dx}\right) h(\Delta_2 x) + b(x, y(x)) h(\Delta_1 x) \cdot h(\Delta_2 x) = \Theta,$$

où $a(x, u)$, $b(x, u)$ sont définies par (1), (2) et $\Delta_1 x$, $\Delta_2 x$ sont des variables, si les conditions (I_1) , (I_2) se ramplacent par les conditions

(I_3) Il existe $r_0 \in \mathbb{C}$ tel que l'égalité $r_0(r_0 - 1)c_0 + r_0 a(c_0) + b_0(c_0) = 0$ soit satisfaite pour tout $c_0 \in Y$ et $2r_0 + n - 1 \neq 0$ ($n = 1, 2, \dots$)

$$(I_4) \quad \frac{\|a_0\|}{|2r_0 + n - 1|} \leq \alpha < 1 \quad (n = 1, 2, \dots).$$

En procédant de la même manière comme pour le Théorème 1 on peut prouver le

Théorème 3. Les conditions (I_3) , (I_4) étant remplies, l'équation $(3')$ admet des solutions ayant la forme (4) et

$$\sum_{n=0, \infty} \|c_n\| \lambda^n < \infty \quad \text{pour } |\lambda| < \varepsilon_0, \lambda \in \mathcal{E}.$$

De même, si $a_0(u) = \alpha_0 u$, $b_0(u) = \beta_0 u$ (Remarque 1, 2) on a aussi le

Théorème 4. Si $r_0(r_0 - 1) + r_0 \alpha_0 + \beta_0 = 0$ et $2r_0 + \alpha_0 + n - 1 \neq 0$ ($n = 1, 2, \dots$), alors l'équation $(3')$ admet des solutions ayant la forme (4) et

$$\sum_{n=0, \infty} \|c_n\| \lambda^n < \infty \quad \text{pour } |\lambda| < \varepsilon_0, \lambda \in \mathcal{E}.$$

4. Équations de Bessel dans des espaces de Banach. Nous donnons un résultat sur les équations de Bessel dans des espaces de Banach qui s'obtient de la même manière que le Théorème 2.

Soit l'équation

$$(3'') \quad h^2(x) \frac{d^2 y(x; x_0, y_0)}{dx^2} + h(x)y'(x; x_0) + (h^2(x) - \nu^2)y(x) = \Theta$$

où $h(x_0) = h(y_0) = 1$.

L'équation déterminante $(7'')$ devient dans ce cas $r^2 - \nu^2 = 0$. En procédant comme dans le cas classique on obtient la solution

$$y_\nu(x) = \left(\sum_{n=0, \infty} (-1)^n \frac{\left(\frac{h(x)}{2} \right)^{2n+\nu}}{\Gamma(n+1)\Gamma(n+\nu+1)} \right) c_0, \quad c_0 \in Y.$$

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ECUAȚII DIFERENȚIALE DE TIP FUCHS, DE ORDINUL DOI, ÎN SPAȚII BANACH

(Rezumat)

Se demonstrează teoremele 1...4 de existență a unor soluții de forma (4) pentru ecuații diferențiale de ordinul 2, cu derivate Fréchet, în spații Banach, de tip Fuchs de forma (3) (derivatele Fréchet sînt fixate pe anumite direcții) și de forma (3') (derivatele sînt considerate pe direcții variind arbitrar).

ON A DYNAMICAL THEOREM OF LIOUVILLE

BY

Q. K. GHORI and A. KARIM

1.1. Introduction. According to a well-known theorem of Liouville [8], the Hamiltonian system of equations of motion of a conservative holonomic dynamical system of order $2n$ can be completely integrated by quadratures if n independent integrals in involution are given. This result has been generalized in various directions. A variant of it for Lagrangian redundant coordinates is due to Arzanyh [1]. Another variant developed by Nadile [6] uses certain linear combinations of momenta as one of the sets of n variables. In a subsequent paper [7], Nadile generalizes his result to nonholonomic systems under restrictions on coefficients in the constraint equations.

This paper adopts a group-theoretic approach to discuss the motion of a holonomic dynamical system defined by canonical equations in the Poincaré-Četaev variables. A generalization is obtained for Liouville's theorem on the integration of the equation of motion when half the integrals are known. In so doing a theorem analogous to Lie's theorem on involution-systems, is needed and established.

We shall be concerned with generalization of a different sort again. Considering a conservative holonomic system, and following a recent series of papers [2]...[5], we determine its position at any time t by n independent variables x_p and the displacements by the group of operators x_p with structure constants C_{pqr} defined by the commutation relations

$$(1.1) \quad (x_p, X_q) = C_{pqr} X_r.$$

Throughout this paper, we employ summation convention over repeated indices having the ranges: $p, q, r = 1, 2, \dots, n$; $i, j = 1, 2, \dots, m$.

If $f(x_1, x_2, \dots, x_n; t)$ is an arbitrary function then its variation in a real displacement of the system is given by

$$(1.2) \quad df = \left(\frac{\partial f}{\partial t} + \eta_p X_p f \right) dt,$$

where the parameters η_p characterize real displacements. Let H be the Hamiltonian of the system expressed in terms of the canonical variables x_p, y_p and the time t , the motion of the system is determined by the equations [3]

$$(1.3) \quad \eta_p = \frac{\partial H}{\partial y_p}, \quad \frac{dy_p}{dt} = -X_p H + C_{qir} \eta_q y_r.$$

2. Involution-Systems. If u and v any two functions of the variables x_p, y_p of class C^2 in some domain of these variables, then Poisson bracket (u, v) is defined by the relation

$$(2.1) \quad (u, v) = \frac{\partial v}{\partial y_p} X_p u - \frac{\partial u}{\partial y_p} X_p v + C_{pqr} y_r \frac{\partial u}{\partial y_q} \frac{\partial v}{\partial y_p}.$$

Let $f_1, f_2, \dots, f_m$ be a set of functions, each of class C^2 in a certain domain of the variables x_p, y_p and t , with the property that the Poisson bracket (f_i, f_j) of any pair is identically zero. Such a set of functions is called an involution-system.

We now prove the following

Theorem 1. *If $u = u(f_1, \dots, f_m)$, $v = v(f_1, \dots, f_m)$ are functions of $f_1, \dots, f_m$ of class C^2 in the range of values of the f 's corresponding to the domain of the x 's, y 's and t , then $(u, v) = 0$.*

The theorem is easily established by direct differentiation. In fact, we have

$$X_p u = \frac{\partial u}{\partial f_i} X_p f_i, \quad \frac{\partial u}{\partial y_p} = \frac{\partial u}{\partial f_i} \frac{\partial f_i}{\partial y_p}$$

with similar relations for v . Substituting these results in (2.1), we get

$$(u, v) = \frac{\partial u}{\partial f_i} \frac{\partial v}{\partial f_j} (f_i, f_j),$$

and this is zero, since the f_i form an involution-system. This completes the proof.

A modified form of this result leads to the following

Theorem 2. *If the functions f_i are in involution, and the equations $u_i = 0$ are consequences of the equations $f_i = 0$, then the functions u_i are in involution.*

Thus we arrive at an extension of Lie's theorem [8] in the Poincaré-Četaev terminology.

3. Generalization of Liouville's Theorem. Let the dynamical system described by the canonical equations (1.3) admit n distinct integrals

$$(3.1) \quad \Phi_p(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n, t) = a_p,$$

where a_p are arbitrary constants, and let the functions Φ_p form an involution-system. We solve equations (3.1) to obtain

$$(3.2) \quad y_p = f_p(x_1, x_2, \dots, x_n, a_1, a_2, \dots, a_n, t).$$

Since the functions $\Phi_p - a_p$ are in involution, it follows by Theorem 2 that the functions $y_p - f_p$ are also in involution, and therefore

$$(y_p - f_p, y_q - f_q) = 0,$$

which is equivalent to the equations

$$(3.3) \quad X_p f_q - X_q f_p = C_{pqr} f_r.$$

Also, from (1.2), (1.3) and (3.3) we have

$$\frac{\partial f_p}{\partial t} = \frac{dy_p}{dt} - \eta_q X_q f_p = -X_p H + C_{qpr} \eta_q y_r + C_{pqr} \eta_q f_r - \eta_q X_p f_q.$$

By virtue of (3.2) and the property $C_{pqr} = -C_{qpr}$, we obtain

$$\frac{\partial f_p}{\partial t} = -X_p H - \frac{\partial H}{\partial y_q} X_p f_q.$$

If H_1 denotes the function H when expressed in terms of $x_1, \dots, x_n, a_1, \dots, a_n$ and t , the last result becomes

$$(3.4) \quad \frac{\partial f_p}{\partial t} = -X_p H_1.$$

To satisfy the relations (3.3) and (3.4) it is sufficient to choose a function $V(x_1, \dots, x_n, a_1, \dots, a_n, t)$ such that

$$(3.5) \quad f_p = X_p V, \quad H_1 = -\frac{\partial V}{\partial t}.$$

The verification of (3.4) is immediate if we recall that $(\partial/\partial t, X_p) = 0$. Again, from (1.1) it follows that

$$X_p f_q - X_q f_p = (X_p X_q - X_q X_p) V = (X_p, X_q) V = C_{pqr} X_r V,$$

which verifies (3.3).

The equations (3.5) together with (1.2) imply that

$$(3.6) \quad (f_p \eta_p - H_1) dt = dV.$$

Thus, the form $(f_p \eta_p - H_1) dt$ is the perfect differential of function V .

Now let A_p be the operators X_p and β_p the parameters η_p corresponding to the a_p 's; and let the symbol d denote the total differential of the function V with respect to all its arguments. Then we have

$$dV = (f_p \eta_p - H_1 + \beta_p A_p V) dt.$$

In this equation we replace the a 's by their values Φ_p 's and the β_p 's by their linear expressions in terms of the Φ_p 's obtained from the relations

$$(3.7) \quad \dot{\Phi}_p = \beta_q A_q a_p.$$

Consequently we have an identity

$$\left(\frac{dV}{dt} - \beta_p A_p V \right) dt = (y_p \eta_p - H) dt,$$

where on the left-hand side of the equation we suppose that in dV/dt and $A_p V$ the quantities $a_1, \dots, a_n$ are replaced by their values $\Phi_1, \dots, \Phi_n$, and $\beta_1, \dots, \beta_n$.

by their values in terms of $\dot{\Phi}_1, \dots, \dot{\Phi}_n$ given by (3.7). Hence the equations of the original dynamical problem are equivalent to the first Pfaff's system of the form

$$\left(-\beta_p A_p V + \frac{dV}{dt}\right) dt.$$

To obtain this system, we put

$$\theta_a = -\beta_p A_p V dt + dV, \quad \theta_\delta = -\beta_p A_p V \delta t + \delta V,$$

where d and δ are symbols of independent increments. We have therefore

$$\delta \theta_a - d \theta_\delta = -\beta_p \delta(A_p V) dt + \beta_p d(A_p V) \delta t.$$

Consequently we obtain

$$\beta_p = 0, \quad d(A_p V) = 0.$$

The first set of these n equations, in conjunction with (3.7), implies relations (3.1). The second set of equations shows that the expressions $A_p V$ are constant throughout the motion, i.e. the equations $A_p V = b_p$, where $b_1, \dots, b_n$ are new arbitrary constants, are integrals of the system.

The results of our discussion can be summed up in the following theorem which gives a generalization of Liouville's theorem [8].

Theorem 3. *Let the dynamical system (1.3) admit n distinct integrals (3.1) where a_p are arbitrary constants and the functions Φ_p are in involution; and let (3.1) be solved to give (3.2).*

Then the substitution of f_p for y_p in the expression $(y_p \eta_p - H) dt$ makes it a perfect differential of some function $V(x_1, \dots, x_n, a_1, \dots, a_n, t)$; and the remaining integrals of the system are (3.8) where b_p are arbitrary constants.

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ASUPRA UNEI TEOREME DE DINAMICĂ A LUI LIOUVILLE

(Rezumat)

Utilizînd procedee grupale se discută mișcarea unui sistem dinamic oonom definit prin ecuațiile canonice în variabilele lui Poincaré-Cetaev. Se generalizează o teoremă a lui Liouville asupra integrării ecuațiilor de mișcare cînd se cunosc jumătate din integralele prime, după ce în prealabil se stabilește o teoremă analogă cu teorema lui Lie asupra sistemelor în involuție.

DISCONTINUITĂȚI INFINITESIMALE ÎN HIDRODINAMICA RADIATIVĂ RELATIVISTĂ

DE

GH. PROCOPIUC

1. Introducere. În înțelegerea anumitor fenomene astrofizice, care se produc în particular în sistemul solar, teoria fluidelor relativiste devine tot mai importantă. În studiul matematic al acestor fluide un loc aparte îl ocupă problema undelor. Propagarea discontinuităților infinitesimale într-un fluid perfect relativist în prezența unui câmp electromagnetic a fost studiată în [1], [2]. În prezenta lucrare se cercetează condițiile în care discontinuități infinitesimale se pot propaga într-un fluid perfect relativist în prezența unui câmp radiativ, definit ca în [3], [4]. Se demonstrează că discontinuitățile câmpului radiativ se propagă în lungul bicaracteristicilor asociate acestor unde.

2. Ecuațiile hidrodinamicii radiative relativiste. Fie V_4 o varietate de dimensiune 4, a cărei metrică hiperbolică normală ds^2 , de semnătură $+- - -$, se exprimă în coordonate locale sub forma

$$(2.1) \quad ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta, \quad (\alpha, \beta = 0, 1, 2, 3).$$

Presupunem că tensorul metric $g_{\alpha\beta}(x^\lambda)$ este de clasă C^1 și pe bucăți de clasă C^2 . Vectorul viteză se definește prin $u^\alpha = dx^\alpha/ds$, ceea ce implică caracterul său unitar

$$(2.2) \quad u^\alpha u_\alpha = 1.$$

Într-un domeniu din V_4 , fluidul perfect în interacțiune cu un câmp radiativ este descris prin tensorul energie-impuls

$$(2.3) \quad U_{\alpha\beta} = r f u_\alpha u_\beta - p g_{\alpha\beta} + R_{\alpha\beta},$$

unde r este densitatea de materie a fluidului, f este indicele fluidului, iar p este presiunea sa. Dacă T este temperatura proprie și η entropia specifică, atunci

$$T d\eta = df - \frac{1}{r} dp.$$

În (2.3), $R_{\alpha\beta}$ este tensorul energie-impuls al câmpului radiativ

$$(2.4) \quad R_{\alpha\beta} = R u_\alpha u_\beta + u_\alpha R_\beta + u_\beta R_\alpha - \frac{1}{3} R g_{\alpha\beta}, \quad \text{cu } u^\alpha R_\alpha = 0,$$

unde R este densitatea de energie radiativă, R_α — fluxul radiativ, iar

$$\frac{1}{g^{\alpha\beta}} = g^{\alpha\beta} - u^\alpha u^\beta$$

este proiectorul spațiului [5].

Sistemul diferențial al hidrodinamicii radiative relativiste este dat de ecuația de conservare a densității de materie

$$(2.5) \quad \nabla_\alpha (ru^\alpha) = 0;$$

ecuațiile de conservare a tensorului energie-impuls

$$\nabla_\alpha U^{\alpha\beta} = 0,$$

care aici sînt echivalente cu ecuațiile

$$(2.6) \quad rTu^\alpha \nabla_\alpha \eta = c^{-1}P,$$

$$(2.7) \quad rf u^\alpha \nabla_\alpha u^\beta - r \frac{1}{g^{\alpha\beta}} (f_{,\alpha} - T\eta_{,\alpha}) = \bar{F}^\beta,$$

unde $P = \sigma c(R - aT^4)$, $\bar{F}_\alpha = \sigma R_\alpha$, și ecuațiile cîmpului radiativ

$$(2.8) \quad u^\alpha \nabla_\alpha R + \nabla_\alpha R^\alpha + \frac{4}{3} R \nabla_\alpha u^\alpha - R^\alpha A_\alpha = -c^{-1}P,$$

$$(2.9) \quad u^\beta \nabla_\beta R_\alpha - \frac{1}{3} \frac{1}{g^{\beta\gamma}} R_{,\beta} + \frac{4}{3} R A_\alpha + R_\alpha \nabla_\beta u^\beta + R^\beta \nabla_\beta u_\alpha + R^\beta A_\beta u_\alpha = -\bar{F}_\alpha,$$

unde $A_\alpha = u^\beta \nabla_\beta u_\alpha$, este vectorul accelerație.

3. Unde radiative. Fie Σ o hipersuprafață regulată într-un domeniu a lui V_4 , de ecuație $\varphi = 0$. Viteza v_Σ a hipersuprafeței Σ în raport cu fluidul, adică în raport cu direcția temporală u , este dată [1] de formula

$$\frac{v_\Sigma^2}{c^2} = - \frac{(u^\alpha l_\alpha)^2}{\frac{1}{g^{\alpha\beta}} l_\alpha l_\beta},$$

unde $l_\alpha = \nabla_\alpha \varphi$.

Într-un domeniu Ω din V_4 în care variabilele dependente f , η , u^α , R , R^α sînt continue, presupunem că primele lor derivate sînt discontinue la traversarea lui Σ . Dacă δ este operatorul de discontinuitate infinitesimală, $[\nabla_\alpha] = l_\alpha \delta$, din ecuațiile (2.5)...(2.9) obținem pentru distribuțiile δf , $\delta \eta$, δu^α , δR , δR^α , sistemul de ecuații

$$(3.1) \quad rl_\alpha \delta u^\alpha + r' u^\alpha l_\alpha \delta f = 0,$$

$$(3.2) \quad u^\alpha l_\alpha \delta \eta = 0,$$

$$(3.3) \quad f u^\alpha l_\alpha \delta u^\beta - \frac{1}{g^{\alpha\beta}} (l_\alpha \delta f - T l_\alpha \delta \eta) = 0,$$

$$(3.4) \quad u^\alpha l_\alpha \delta R + l_\alpha \delta R^\alpha + \frac{4}{3} R l_\alpha \delta u^\alpha - u^\alpha R^\beta l_\beta \delta u_\beta = 0,$$

$$(3.5) \quad u^\beta l_\beta \delta R_\alpha - \frac{1}{3} \frac{1}{g^{\beta\gamma}} l_\beta \delta R + \frac{4}{3} R u^\beta l_\beta \delta u_\alpha + R_\alpha l_\beta \delta u^\beta + R^\beta l_\beta \delta u_\alpha + R^\beta u_\alpha l_\beta \delta u_\beta = 0.$$

Să cercetăm în ce condiții distribuțiile de mai sus nu se anulează simultan. Din (3.2), pentru $u^\alpha l_\alpha = 0$, $\delta \eta$ poate fi nenulă. Obținem astfel

hipersuprafețele generate de liniile de curent, numite *unde de entropie*. Viteza lor în raport cu fluidul este nulă. Pentru discontinuitățile corespunzătoare avem

$$l_\alpha \delta u^\alpha = 0, \quad \delta f - T \delta \eta = 0, \quad l_\alpha \delta R^\alpha = 0, \quad -\frac{1}{3} l_\alpha \delta R + l^\beta R_\beta \delta u_\alpha = 0.$$

Presupunem acum $u^\alpha l_\alpha \neq 0$; atunci din (3.2) urmează

$$(3.2') \quad \delta \eta = 0$$

și deci din (3.3) obținem

$$(3.3') \quad f u^\beta l_\beta l_\alpha \delta u^\alpha - \frac{1}{g^{\alpha\beta}} l_\alpha l_\beta \delta f = 0,$$

care împreună cu (3.1) formează un sistem liniar omogen în $l_\alpha \delta u^\alpha$ și δf , cu soluții nebanale, dacă

$$(3.6) \quad H(l) = \frac{1}{g^{\alpha\beta}} l_\alpha l_\beta + (u^\alpha l_\alpha)^2$$

este nul, unde $\gamma = f' r_f / r$. Ecuația $H(l) = 0$ este ecuația undelor sonice ale fluidului, cu viteza dată de

$$\frac{v_H^2}{c^2} = \frac{1}{\gamma},$$

cu $\gamma > 1$. Apoi, din (3.4) și (3.5), ținând seama de (3.1) și (3.3') și de ecuația

$$\delta u^\beta = \frac{1}{g^{\alpha\beta}} l_\alpha \delta f, \quad f u^\alpha l_\alpha$$

rezultă

$$(3.7) \quad \begin{cases} u^\beta l_\beta \delta R + l^\alpha \delta R_\alpha \simeq 0, \\ -\frac{1}{3} \frac{1}{g^{\alpha\beta}} l_\alpha l_\beta \delta R + u^\beta l_\beta l^\alpha \delta R_\alpha \simeq 0, \end{cases}$$

unde $\simeq 0$ desemnează termeni proporționali cu δf .

Determinantul acestui sistem liniar în necunoscutele δR și $l^\alpha \delta R_\alpha$ se scrie

$$(3.8) \quad R(l) = \frac{1}{3} \frac{1}{g^{\alpha\beta}} l_\alpha l_\beta + (u^\alpha l_\alpha)^2.$$

Hipersuprafețele soluții ale ecuației $R(l) = 0$ sînt numite *unde radiative*. Viteza lor în raport cu fluidul este dată de

$$\frac{v_R^2}{c^2} = \frac{1}{3}.$$

4. Proprietăți ale razelor asociate undelor radiative. Fie φ o soluție a ecuației undelor radiative $R(l) = 0$. Presupunem că pentru undele considerate $H(l) \neq 0$, deci

$$(4.1) \quad \delta f = 0, \quad \delta u^\alpha = 0.$$

În acest caz (3.7) devine un sistem liniar omogen care admite soluții nebanale

$$(4.2) \quad l^\beta \delta R_\beta = \frac{1}{3} \frac{g^{\alpha\beta} l_\alpha l_\beta}{u^\alpha l_\alpha} \delta R,$$

iar din (3.5) rezultă

$$(4.3) \quad \delta R_\beta = \frac{1}{3} \frac{g^{\alpha\beta} l_\alpha}{u^\alpha l_\alpha} \delta R.$$

Generatoarea de contact cu conul $R(l)=0$ este definită de vectorul

$$(4.4) \quad N^\beta = \frac{1}{2} \frac{\partial R(l)}{\partial l_\beta} = r^{\alpha\beta} l_\alpha,$$

unde am notat

$$(4.5) \quad r^{\alpha\beta} = \frac{1}{3} \frac{1}{g^{\alpha\beta}} + u^\alpha u^\beta.$$

Traectoriile cîmpului de vectori N^β sînt bicaracteristicile sau razele asociate acestor unde. Ne propunem să arătăm că δR și deci și δR_β se propagă în lungul razelor, adică δR verifică un sistem diferențial de forma

$$(4.6) \quad N^\beta \nabla_\beta \delta R \approx 0,$$

unde ≈ 0 înseamnă termeni proporționali cu δR .

Presupunem că într-un domeniu Ω , funcțiile $f, \eta, u^\alpha, R, R^\alpha$ sînt continue. Dacă Ω_1 și Ω_2 sînt domeniile în care hipersuprafața Σ împarte pe Ω , presupunem că $f, \eta, u^\alpha, R, R^\alpha$ sînt de clasă C^2 pe fiecare din aceste domenii și că derivatele lor prime și secunde converg uniform spre funcții cu valori definite pe Σ , cînd φ tinde la zero. În aceste condiții există distribuțiile $\bar{f}, \bar{\eta}, \bar{u}^\alpha, \bar{R}, \bar{R}^\alpha$ astfel încît să avem

$$(4.7) \quad [\nabla_\alpha \nabla_\beta f] = \nabla_\alpha \nabla_\beta \delta f + l_\alpha \nabla_\beta \delta f + l_\beta \nabla_\alpha \delta f + l_\alpha l_\beta \bar{f}, \quad \text{etc.}$$

Ecuția (2.5) se mai scrie

$$\nabla_\alpha u^\alpha + \frac{\gamma}{f} u^\alpha \nabla_\alpha f + \frac{r'_\eta P}{r^2 c T} = 0,$$

de unde, prin derivare, obținem

$$\nabla_\beta \nabla_\alpha u^\alpha + \frac{\gamma}{f} u^\alpha \nabla_\beta \nabla_\alpha f + \nabla_\beta \left(\frac{\gamma}{f} u^\alpha \right) \nabla_\alpha f + \nabla_\beta \left(\frac{r'_\eta P}{r^2 c T} \right) = 0.$$

Scriem această relație de o parte și de alta a lui Σ și le scădem. Obținem astfel, ținînd seama de (3.2') și (4.1):

$$(4.8) \quad [\nabla_\beta \nabla_\alpha u^\alpha] + \frac{\gamma}{f} u^\alpha [\nabla_\beta \nabla_\alpha f] \approx 0,$$

unde ≈ 0 desemnează termeni proporționali cu δR .

În mod analog, din (2.6), urmează

$$u^\alpha [\nabla_\beta \nabla_\alpha \eta] \approx 0$$

sau, ținând seama de (4.7) și (4.1): $(u^\alpha l_\alpha) l_\beta \bar{\eta} \approx 0$, adică $\bar{\eta} \approx 0$, deci

$$(4.9) \quad [\nabla_\beta \nabla_\alpha \eta] \approx 0.$$

Tot astfel, din (2.8), avem

$$(4.10) \quad f u^\alpha [\nabla_\beta \nabla_\alpha u^\beta] - g^{\alpha\beta} [\nabla_\beta \nabla_\alpha f] \approx 0.$$

Dar, din (4.7), cu (4.1), obținem

$$(4.11) \quad [\nabla_\alpha \nabla_\beta u^\lambda] = l_\alpha l_\beta \bar{u}^\lambda, \quad [\nabla_\alpha \nabla_\beta f] = l_\alpha l_\beta \bar{f},$$

care înlocuite în (4.8) și (4.10), conduc la sistemul liniar neomogen în $l_\beta \bar{u}^\beta$ și $\bar{f}$:

$$(4.12) \quad \begin{cases} l_\beta \bar{u}^\beta + \frac{\gamma}{f} u^\alpha l_\alpha \bar{f} \approx 0 \\ f l_\beta \bar{u}^\beta - g^{\alpha\beta} l_\alpha l_\beta \bar{f} \approx 0, \end{cases}$$

al cărui determinant este $H(l) \neq 0$. Deci, putem scrie în mod unic

$$(4.13) \quad l_\beta \bar{u}^\beta \approx 0, \quad \bar{f} \approx 0 \quad \text{și deci} \quad \bar{u}^\beta \approx 0,$$

$$(4.13) \quad [\nabla_\alpha \nabla_\beta u^\lambda] \approx 0, \quad [\nabla_\alpha \nabla_\beta f] \approx 0.$$

Din (2.8) și (2.9), ținând seama de (4.1) și (4.3), avem însă

$$(4.14) \quad u^\alpha [\nabla_\beta \nabla_\alpha R] + [\nabla_\beta \nabla_\alpha R^\alpha] + \frac{4}{3} R [\nabla_\beta \nabla_\alpha u^\alpha] - R^\gamma u^\alpha [\nabla_\beta \nabla_\alpha u_\gamma] \approx 0,$$

$$(4.15) \quad u^\beta [\nabla_\alpha \nabla_\beta R^\alpha] - \frac{1}{3} g^{\alpha\beta} [\nabla_\alpha \nabla_\beta R] + \frac{4}{3} R u^\beta [\nabla_\alpha \nabla_\beta u^\alpha] + R^\alpha [\nabla_\alpha \nabla_\beta u^\beta] + \\ + R^\beta [\nabla_\alpha \nabla_\beta u^\alpha] + R^\lambda u^\alpha u^\beta [\nabla_\beta \nabla_\alpha u_\lambda] \approx 0,$$

de unde, eliminând pe $[\nabla_\beta \nabla_\alpha R^\alpha]$ și ținând seama de (4.13), avem

$$r^{\alpha\beta} [\nabla_\beta \nabla_\alpha R] \approx 0,$$

sau, cu (4.7)

$$r^{\alpha\beta} (l_\alpha \nabla_\beta \delta R + l_\beta \nabla_\alpha \delta R) + r^{\alpha\beta} l_\alpha l_\beta \bar{R} \approx 0;$$

dar $r^{\alpha\beta} l_\alpha l_\beta = R(l) = 0$ și deci $r^{\alpha\beta} l_\alpha l_\beta \bar{R} \approx 0$, adică cu (4.4),

$$N^\beta \nabla_\beta \delta R \approx 0.$$

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INFINITESIMAL DISCONTINUITIES IN RADIATIVE RELATIVISTIC
HYDRODYNAMICS

(Summary)

We study the propagation of the infinitesimal discontinuities in a perfect relativistic fluid in the presence of a radiation field. We prove that the discontinuities of the radiation field propagate along bicharacteristics associated with these waves.

PLANE BOUNDARY-VALUE PROBLEMS IN THE NONLINEAR GENERALIZED THEORY OF RODS

II. THE RESOLUTION OF A BOUNDARY-VALUE PROBLEM BY ITERATIVE NEWTON'S METHOD

BY

GH. CHIORESCU

1. Introduction. There is very few methods for the study of the nonlinear problems. In some previous paper the variational calculus [1] and the theory of monotone operators [2], [3], [4] were used. In the present paper we solve a boundary-value problem by an iterative type Newton's method.

2. Newton's Method in Nonlinear Boundary-Value Problem. The iterative technique presented in this section following the papers [5] and [6], is applied directly to the operator that results from the boundary value problem; no formulation as an integral equation is required, and from this point of view, differs from the traditional works.

Let $C[a, b]$ denote the linear space of continuous functions from the compact interval $[a, b]$ into n -dimensional real arithmetic space R_n , and let $C^1[a, b]$ be the subspace of continuously differentiable functions on $[a, b]$. The spaces $C[a, b]$ and $C^1[a, b]$ will be made into some Banach spaces with the norms, respectively

$$\|x\|_0 = \sup_{S \in [a, b]} \|x(S)\|, \quad x \in C[a, b];$$

$$\|x\|_1 = \|x\|_0 + \|x'\|_0, \quad x \in C^1[a, b].$$

We also consider the product space $Y = C[a, b] \times R_m$, which is a Banach space under the norm

$$\|\psi, v\| = \max \{\|\psi\|_0, \|v\|\}, \quad [\psi, v] \in Y.$$

We shall consider the boundary value problem for a first order system of n ordinary differential equation on $[a, b]$ given by

$$(2.1) \quad x' + F(x, S) = 0, \quad f(x) = \sum_{i=1}^{\infty} B_i x(S_i) - c = 0,$$

where $c \in R_m$, $\{B_i\}$ is a sequence of real $m \times n$ matrices such that $\sum \|B_i\| < \infty$, and $S_i \in [a, b]$ for $i=1, 2, \dots$. The function f is a mapping from a subset of $C^1[a, b]$ into R_m , where m and n are not necessarily equal. We shall assume that the function F in (2.1) is at least continuously differentiable on $U \times$

$\times [a, b]$ where U is an open subset of R_n . The domain of the operator $\mathbf{f}$ is assumed to be an open subset D of $C^1[a, b]$, and we shall require that $\mathbf{x}(S) \in U$, $S \in [a, b]$, for every choice of $\mathbf{x} \in D$. Hence, we can define an operator $\mathbf{T}$ on D by

$$\mathbf{T}(\mathbf{x})(S) = \mathbf{F}(\mathbf{x}(S), S), \quad a \leq S \leq b.$$

Since $\mathbf{F}$ is continuously differentiable it follows that $\mathbf{T}$ maps D into $C[a, b]$ and that $\mathbf{T}$ is continuously Fréchet differentiable on D . We define the operator $\mathbf{P}: D \rightarrow Y$ by

$$(2.2) \quad \mathbf{P}(\mathbf{x}) = \left[\mathbf{x}' + \mathbf{T}(\mathbf{x}), \sum_1^\infty \mathbf{B}_i \mathbf{x}(S_i) - \mathbf{c} \right].$$

A function $\mathbf{x} \in C^1[a, b]$ is a solution of (2.1) if and only if it is a solution of $\mathbf{P}(\mathbf{x}) = 0$.

Theorem 2.1. *We assume the operator $\mathbf{T}$ twice continuously Fréchet differentiable on D . For $\mathbf{x}_0 \in D$ let Φ be a fundamental matrix on $[a, b]$ or*

$$\mathbf{x}' + \dot{\mathbf{T}}(\mathbf{x}_0) = 0$$

and define

$$K_1 = \sup_{S \in [a, b]} \|\Phi(S)\|, \quad K_2 = \sup_{S \in [a, b]} \|\Phi(S)^{-1}\|.$$

Let α, β, K be positive constants and let $\mathbf{M}^+$ a $n \times m$ matrix such that

$$(i) \quad \|\mathbf{P}(\mathbf{x}_0)\| \leq \alpha;$$

$$(ii) \quad \mathbf{M}\mathbf{M}^+ = \mathbf{E}_m \text{ where } \mathbf{M} = \sum \mathbf{B}_i \Phi_i(S_i);$$

$$(2.3) \quad (iii) \quad 1 + (1 + \sup \|\dot{\mathbf{T}}(\mathbf{x}_0)(S)\| + K_1\{\|\mathbf{M}^+\| + K_1 K_2 \|\mathbf{M}^+\|(b-a) \sum \|\mathbf{B}_i\| + K_2(b-a)\}) \leq \beta;$$

$$(iv) \quad \|\ddot{\mathbf{P}}(\mathbf{x})\| \leq K \text{ for all } \mathbf{x} \in \sum (\mathbf{x}_0, r_0), \text{ where } \overline{\sum} (\mathbf{x}_0, r_0) \subset D \text{ and } \ddot{\mathbf{P}}(\mathbf{x})\mathbf{z}_1\mathbf{z}_2 = \ddot{\mathbf{T}}(\mathbf{x})\mathbf{z}_1\mathbf{z}_2, 0; \mathbf{z}_1, \mathbf{z}_2 \in C^1[a, b]$$

$$r_0 = \frac{1 - (1 - 2h)^{1/2}}{\beta K}, \quad h = \alpha \beta^2 K \leq \frac{1}{2}.$$

Conclusion. *The Newton iteration for $\mathbf{P}$ starting at $\mathbf{x}_0$, namely*

$$\mathbf{x}_{k+1} = \mathbf{x}_k - [\dot{\mathbf{P}}(\mathbf{x}_k)^+] \mathbf{P}(\mathbf{x}_k), \quad k = 0, 1, \dots$$

yields a sequence $\{\mathbf{x}_k\}$ which is contained in $\overline{\sum} (\mathbf{x}_0, r_0)$ and converges to an element $\mathbf{x}^ \in \overline{\sum} (\mathbf{x}_0, r_0)$ such that $\mathbf{P}(\mathbf{x}^*) = 0$. For any nonnegative integer k , an error estimate is given by*

$$\|\mathbf{x}^* - \mathbf{x}_k\|_1 \leq \frac{1}{\beta K} \frac{(2h)^{2^k}}{2^k}.$$

Further if $h < \frac{1}{2}$ and (2.3) holds in the open ball $\sum (\mathbf{x}_0, r_1)$ where

$$r_1 = \frac{1 + (1 - 2h)^{1/2}}{\beta K}, \quad \sum (\mathbf{x}_0, r_1) \subset D,$$

and $\mathbf{M}^+$ is a $n \times m$ invertible matrix, then $\mathbf{x}^*$ the unique solution of $\mathbf{P}(\mathbf{x}) = 0$ in $\Sigma(\mathbf{x}_0, \mathbf{r}_1)$. If $h = \frac{1}{2}$, then $\mathbf{x}^*$ is the unique solution of $\mathbf{P}(\mathbf{x}) = 0$ in $\Sigma(\mathbf{x}_0, \mathbf{r}_0)$ provided $\mathbf{M}^+$ is an $n \times m$ invertible matrix.

3. Formulation of a Boundary Value Problem. First we make the following hypothesis:

(H₁) the rod is straight, such that we study longitudinal extension and transversal contraction ($\beta = \Phi = 0$ or $\eta = 0$ and $\xi = \varepsilon$);

(H₂) all the components of the body forces are zero except of f^3 and p^{11} which are constants.

The geometrical equations in this case are given by

$$(3.1) \quad \mathbf{d}_1 = (1 + \rho)\mathbf{a}_1, \quad \mathbf{d}_2 = \mathbf{a}_2, \quad \mathbf{d}_3 = \mathbf{a}_3,$$

$$(3.2) \quad \gamma_{11} = (1 + \rho)^2 - 1, \quad \gamma_{13} = 0, \quad \gamma_{33} = (1 + \varepsilon)^2 - 1, \quad k_{11} = \rho'(1 + \rho), \quad k_{13} = 0,$$

$$(3.3) \quad \mathbf{d}^1 = \frac{1}{1 + \rho} \mathbf{a}_1, \quad \mathbf{d}^2 = \mathbf{a}_2, \quad \mathbf{d}^3 = \frac{\mathbf{a}_3}{(1 + \varepsilon)^2},$$

$$(3.4) \quad \mathbf{a}^1 = \mathbf{a}_1, \quad \mathbf{a}^2 = \mathbf{a}_2, \quad \mathbf{a}^3 = \frac{\mathbf{a}_3}{(1 + \varepsilon)^2},$$

$$(3.5) \quad \lambda_1^1 = \frac{\rho'}{1 + \rho}, \quad \lambda_1^3 = 0.$$

We further suppose

$$(3.6) \quad \bar{\pi}^1 = p^{13} = 0.$$

Consequently, the equations (3.12) and (3.14) (Part I), become

$$(3.7) \quad \bar{\pi} = 2\rho_0 \frac{\partial \psi^*}{\partial \gamma_{33}}, \quad \bar{\pi}^{11} = 2\rho_0 \frac{\partial \psi^*}{\partial \gamma_{11}}, \quad p^{11} = \rho_0(1 + \rho) \frac{\partial \psi^*}{\partial k_{11}},$$

$$(3.8) \quad \bar{\pi} = n^3, \quad \bar{\pi}^1 = \frac{2}{1 + \rho} n^1, \quad \bar{\pi}^{11} = \frac{1}{1 + \rho} \left(\pi^{11} - \frac{\rho'}{1 + \rho} p^{11} \right).$$

Obviously, we have also

$$(3.9) \quad n^1 = \pi^{13} = 0.$$

Such a way, only the following two equilibrium equations are not identically verified

$$(3.10) \quad n^{3'} + n^3 \frac{\varepsilon'}{1 + \varepsilon} + \rho_0 f^3 = 0, \quad p^{11'} + \rho_0 l^{11} = \pi^{11},$$

where

$$(3.11) \quad \pi^{11} = 2\rho_0(1 + \rho) \frac{\partial \psi^*}{\partial \gamma_{11}} + \rho_0 \rho' \frac{\partial \psi^*}{\partial k_{11}}, \quad n^3 = 2\rho_0 \frac{\partial \psi^*}{\partial \gamma_{33}},$$

$$p^{11} = \rho_0(1 + \rho) \frac{\partial \psi^*}{\partial k_{11}}.$$

We assume the following boundary conditions

$$(3.12) \quad s(a)=a, \quad s(b)=b+\Lambda, \quad \rho(a)=c_1, \quad \rho(b)=c_2.$$

By using (3.11), the equations (3.10) can be written

$$(3.13) \quad \begin{aligned} \varepsilon'[4(1+\varepsilon)\alpha_1] + \rho''[2(1+\varepsilon)\alpha_2] + 2\rho'^2\alpha_2 + 4\rho'(1+\rho)\alpha_3 + 2\left(1 + \frac{\rho'_0}{\rho_0}\right)\alpha_4 + f^3 = 0, \\ \varepsilon'[2(1+\varepsilon)\alpha_2] + \rho''[(1+\rho)\alpha_5] + \rho'^2\alpha_5 + 2\rho'(1+\rho)\alpha_6 + \frac{\rho'_0}{\rho_0}(1+\rho)\alpha_7 - \\ - 2(1+\rho)\alpha_8 + l^{11} = 0, \end{aligned}$$

where

$$(3.14) \quad \begin{aligned} \alpha_1 &= \frac{\partial^2 \psi^*}{\partial \gamma_{33} \partial \gamma_{33}}, \quad \alpha_2 = \frac{\partial^2 \psi^*}{\partial \gamma_{33} \partial k_{11}}, \quad \alpha_3 = \frac{\partial^2 \psi^*}{\partial \gamma_{33} \partial \gamma_{11}}, \quad \alpha_4 = \frac{\partial^2 \psi^*}{\partial \gamma_{33}}, \\ \alpha_5 &= \frac{\partial^2 \psi^*}{\partial k_{11} \partial k_{11}}, \quad \alpha_6 = \frac{\partial^2 \psi^*}{\partial k_{11} \partial \gamma_{11}}, \quad \alpha_7 = \frac{\partial^2 \psi^*}{\partial k_{11}}, \quad \alpha_8 = \frac{\partial^2 \psi^*}{\partial \gamma_{11}}. \end{aligned}$$

We make now the following hypothesis:

(H₃) The strain energy function ψ^* is strictly convex in the variables γ_{33} and k_{11} for fixed values of the remaining variables.

Since ψ^* is assumed to be twice continuously differentiable, the Hessian matrix of second derivatives of ψ^* with respect to γ_{33} and k_{11} is positive definite, i.e.

$$(3.15) \quad \Delta = \frac{\partial^2 \psi^*}{\partial \gamma_{33} \partial \gamma_{33}} \frac{\partial^2 \psi^*}{\partial k_{11} \partial k_{11}} - \left(\frac{\partial^2 \psi^*}{\partial \gamma_{33} \partial k_{11}} \right)^2 > 0.$$

In view of (3.15), the system (3.13) may be solved with respect to ε and ρ'' , and we obtain

$$(3.16) \quad \varepsilon' = \beta_1 + \beta_2 f^3 + \beta_3 l^{11}, \quad \rho'' = \beta_4 + \beta_5 f^3 + \beta_6 l^{11},$$

where

$$(3.17) \quad \begin{aligned} \beta_1 &= \left\{ -4\rho'(1+\rho)^2\alpha_6\alpha_3 - (1+\rho)\left(2\frac{\rho'_0}{\rho_0} + 2\right)\alpha_6\alpha_4 + 2(1+\rho)\rho'^2\alpha_2\alpha_5 + \right. \\ &\quad \left. + 2\rho'(1+\rho)^2\alpha_2\alpha_6 + 2\frac{\rho'_0}{\rho_0}(1+\rho)^2\alpha_2\alpha_4 - 4(1+\rho)^2\alpha_2\alpha_8 \right\} \frac{1}{4(1+\varepsilon)(1+\rho)\Delta}, \\ \beta_2 &= -\frac{\alpha_6}{4(1+\varepsilon)\Delta}, \quad \beta_3 = \frac{2\alpha_2}{4(1+\varepsilon)\Delta}, \\ \beta_4 &= \left\{ -4\rho'^2(1+\varepsilon)\alpha_1\alpha_5 - 8\rho'(1+\varepsilon)(1+\rho)\alpha_1\alpha_6 - 4\frac{\rho'_0}{\rho_0}(1+\varepsilon)(1+\rho)\alpha_1\alpha_7 + \right. \\ &\quad \left. + 8(1+\varepsilon)(1+\rho)\alpha_1\alpha_8 + 4\rho'^2(1+\varepsilon)\alpha_2^2 + 8\rho'(1+\varepsilon)(1+\rho)\alpha_2\alpha_3 + \right. \\ &\quad \left. + 2(1+\varepsilon)\left(2\frac{\rho'_0}{\rho_0} + 2\right)\alpha_2\alpha_4 \right\} \frac{1}{4(1+\varepsilon)(1+\rho)\Delta}, \\ \beta_5 &= \frac{\alpha_2}{2(1+\rho)\Delta}, \quad \beta_6 = -\frac{\alpha_1}{(1+\rho)\Delta}. \end{aligned}$$

The boundary-value problem (3.12)...(3.16) may be written in the form

$$(3.18) \quad \mathbf{x}' + \mathbf{F}(\mathbf{x}, S) = 0,$$

$$(3.19) \quad \mathbf{B}^1 \mathbf{x}(a) + \mathbf{B}^2 \mathbf{x}(b) - \mathbf{c} = 0,$$

where

$$\mathbf{x} = (x^1, x^2, x^3, x^4) = (s - S, \varepsilon, \rho, \rho'),$$

$$\mathbf{F} = (-x^2, -\beta_1(\mathbf{x}) - \beta_2(\mathbf{x})f^3(S) - \beta_3(\mathbf{x})l^{11}(S), -x^4, -\beta_4(\mathbf{x}) - \beta_5(\mathbf{x})f^3 - \beta_6(\mathbf{x})l^{11}),$$

$$\mathbf{B}^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B}^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{c} = (0, \Lambda, c_1, c_2).$$

4. The Resolution of the Boundary Value Problem. We are concerning with classical solution, so that $\mathbf{x} \in C^1[a, b]$. Let U be the set

$$U = [0, \Lambda] \times (-1, \infty) \times (-1, \infty) \times (-\infty, \infty)$$

and we suppose $\mathbf{x}(S) \in U$, $S \in [a, b]$, for every $\mathbf{x} \in D$, $D \subset C^1[a, b]$. The set U is a consequence of the mechanical conditions $1 + \varepsilon > 0$ and $1 + \rho > 0$.

We define the operator $\mathbf{T}$ on D by

$$\mathbf{T}(\mathbf{x})(S) = \mathbf{F}(\mathbf{x}, S).$$

A function $\mathbf{x} \in C_1[a, b]$ is a solution of boundary value problem (3.18), (3.19) if and only if is a solution of equation

$$(4.1) \quad \mathcal{Q}(\mathbf{x}) = 0,$$

where

$$\mathcal{Q}(\mathbf{x}) = [\mathbf{x}' + \mathbf{T}(\mathbf{x}), \mathbf{B}^1 \mathbf{x}(a) + \mathbf{B}^2 \mathbf{x}(b) - \mathbf{c}].$$

Furthermore we make now the following hypotheses:

(H₄) the energy strain function ψ^* is four continuously differentiable in respect to its arguments;

(H₅) the mass density in the reference configuration is continuously differentiable on $[a, b]$;

(H₆) the body forces are continuous functions on $[a, b]$.

The Fréchet derivative of $\mathcal{Q}(\mathbf{x})$ at $\mathbf{x} = 0$ (corresponding to reference configuration) is given by

$$\dot{\mathcal{Q}}(0)\mathbf{x} = [\mathbf{x}' + \mathbf{T}(0)\mathbf{x}, \mathbf{B}^1 \mathbf{x}(a) + \mathbf{B}^2 \mathbf{x}(b)].$$

From all the previous hypotheses (H₁) - (H₆) it results that the linear system

$$(4.2) \quad \mathbf{x}' + \dot{\mathbf{T}}(0)\mathbf{x} = 0$$

has solution on $[a, b]$ ([9], Theorem 4.1, Chap. III).

Let Φ be a such fundamental matrix on $[a, b]$ and let K_1 , K_2 be the real numbers

$$(4.3) \quad K_1 = \sup_{S \in [a, b]} \|\Phi(S)\|, \quad K_2 = \sup_{S \in [a, b]} \|\Phi^{-1}(S)\|.$$

Let also $\mathbf{M}$ be the matrix given by

$$\mathbf{M} = \mathbf{B}^1 \Phi(a) + \mathbf{B}^2 \Phi(b).$$

We suppose Φ to be such that

(H₇) the matrix $\mathbf{M}$ is invertible.

If the initial stresses are absent then we obtain

$$(4.4) \quad \mathcal{Q}(0) = [(0, -\beta_2(0)f^3(S) - \beta_3(0)l^{11}, 0, -\beta_5(0)f^3 - \beta_6(0)l^{11}), -c],$$

and (Sec. 2)

$$(4.5) \quad \|\mathcal{Q}(0)\| = \max_{S \in [a, b]} \{ \sqrt{(\beta_2(0)f^3 + \beta_3(0)l^{11})^2 + (\beta_5(0)f^3 + \beta_6(0)l^{11})^2} \sqrt{\Lambda^2 + c_1^2 + c_2^2} \} = \alpha.$$

The Fréchet derivative of the operator $\mathbf{T}(\mathbf{x})$ is given by [7, p. 26]

$$(4.6) \quad \dot{\mathbf{T}}(0) = \frac{\partial \mathbf{F}(0, S)}{\partial \mathbf{x}}.$$

In view of the regularity assumptions given above we can write the inequality

$$(4.7) \quad \|\dot{\mathbf{T}}(0)\| \leq \max \left(1, \sum_{i=1}^4 \left| \sum_{j=1}^3 \frac{\partial \beta_j(0)}{\partial x_i} g_j \right|, \sum_{i=1}^4 \left| \sum_{j=1}^3 \frac{\partial \beta_{j+3}(0)}{\partial x_j} g_j \right| \right) = \beta$$

where $\mathbf{g}$ is the vector

$$(4.8) \quad \mathbf{g} = (l, f^3, l^{11})$$

and for the norm of a matrix we have used

$$\|\mathbf{L}\| = \sup_i \sum_{k=1}^n |a_{ik}|.$$

Also, conforming to hypotheses (H₁)–(H₆) there exists a constant K such that

$$(4.9) \quad \|\ddot{\mathcal{Q}}(\mathbf{x})\| = \|\ddot{\mathbf{T}}(\mathbf{x})\| \leq K \quad \text{for} \quad \|\mathbf{x}\| \leq 2\alpha\beta^2.$$

From all the above results we finally obtain

Theorem. *If the hypotheses (H₁)–(H₇) holds and if the data of the problem (the constitutive assumptions, the reference configuration, the body forces, the boundary conditions) are such that*

$$(4.10) \quad \max_{S \in [a, b]} \left\{ \sqrt{(\beta_2(0)f^3 + \beta_3(0)l^{11})^2 + (\beta_5(0)f^3 + \beta_6(0)l^{11})^2} \sqrt{\Lambda^2 + c_1^2 + c_2^2} \right\} \left\{ 1 + \left[1 + \max \left(1, \sup_{S \in [a, b]} \left(\sum_{i=1}^4 \left| \sum_{j=1}^3 \frac{\partial \beta_j(0)}{\partial x_i} g_j \right| \right) \right. \right. \right. \\ \left. \left. \left. \sum_{i=1}^4 \left| \sum_{j=1}^3 \frac{\partial \beta_{j+3}(0)}{\partial x_j} g_j \right| \right) \right] K_1 [\|\mathbf{M}^{-1}\| + K_1 K_2 \|\mathbf{M}^{-1}\| (b-a)^2 + \right. \\ \left. \left. + K_2 (b-a) \right] \right\}^2 K \leq \frac{1}{2}, \quad \text{or} \quad \alpha \beta^2 K \leq \frac{1}{2}$$

than the iteration

$$(4.11) \quad \mathbf{x}_{k+1} = \mathbf{x}_k - [\mathcal{R}(\mathbf{x}_k)]^{-1} \mathcal{R}(\mathbf{x}_k), \quad k = 0, 1, 2, \dots$$

is a convergent sequence in $\sum (0, r_0)$ where

$$(4.12) \quad r_0 = \frac{1 - (1 - 2\alpha\beta^2 K)^{1/2}}{\beta K},$$

to $\mathbf{x}^*$ the solution of the equation (4.1). An error estimate is given by

$$(4.13) \quad \|\mathbf{x}^* - \mathbf{x}_k\| \leq \frac{1}{\beta K} \frac{(2\alpha\beta^2 K)^{2^k}}{2^k}.$$

Further if (4.10) holds with strict inequality then the solution is unique in $\sum (0, r_1)$ where r_1 is given by

$$(4.14) \quad r_1 = \frac{1 + (1 - 2\alpha\beta^2 K)^{1/2}}{\beta K}.$$

If $\alpha\beta^2 K = \frac{1}{2}$ then $\mathbf{x}^*$ is the unique solution of (4.1) in $\sum (0, r_0)$.

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PROBLEME LA LIMITĂ PLANE ÎN TEORIA NELINIARĂ GENERALIZATĂ A BARELOR

II. Rezolvarea unei probleme la limită prin procedeul iterativ al lui Newton

(Rezumat)

Folosind ecuațiile de echilibru deduse în Partea I-a, se formulează o problemă la limită în deplasări. Pentru rezolvarea ei s-a folosit o metodă de aproximare de tip Newton, care spre deosebire de toate metodele clasice de acest tip, aplică procedeul iterativ direct la operatorul ce rezultă din problema la limită. În final sînt impuse condiții asupra datelor problemei pentru ca șirul aproximantelor să convergă la soluția problemei.

A VARIATIONAL FORMULATION OF THE FLOW EQUATIONS OF MICROPOLAR FLUIDS

BY

VALERIU A. SAVA

1. Introduction. Variational formulations of physical problems are well-known for their simplicity and elegance. In this paper we explore the possibility of applying variational methods to the flow equations of micropolar fluids. Deshpande [1] has shown that the structure of the Navier-Stokes equations is such that it cannot get a single functional whose extremization yields the equations. Instead he obtain two harmonic functionals with a close relation to the Navier-Stokes equations. We find that the flow equations of a micropolar fluid can be obtained with the help of three functionals.

2. Two-dimensional Flows. The governing partial differential equations for two-dimensional flow of incompressible micropolar fluids are [2]

$$(1) \quad \begin{aligned} u_x + v_y &= 0, \\ uu_x + vv_y &= -\frac{1}{\rho} p_x + (\nu + \kappa)(u_{xx} + u_{yy}) + \kappa \varphi_y, \end{aligned}$$

$$(2) \quad uv_x + vv_y = -\frac{1}{\rho} p_y + (\nu + \kappa)(v_{xx} + v_{yy}) - \kappa \varphi_x,$$

$$a^2(u\varphi_x + v\varphi_y) = \gamma(\varphi_{xx} + \varphi_{yy}) + \kappa(v_x - u_y) - 2\kappa\varphi,$$

where u, v are components of the velocity, φ is the component of microrotation vector, p the hydrostatic pressure, ν, κ, γ are the kinematic viscosity coefficients, and the subscripts denote partial differentiation with respect to concerned variable and this notation is followed throughout the paper.

We consider flows in a region D exterior to the closed boundary B of the body. The boundary conditions will be

$$u=0, \quad v=0, \quad \varphi=0 \quad \text{on } B.$$

We now consider the following problem: does there exists a functional $A(u, v, \varphi, p)$ depending on the functions $u(x, y)$, $v(x, y)$, $\varphi(x, y)$ and $p(x, y)$ such that its functional derivatives $\delta A/\delta p$, $\delta A/\delta u$, $\delta A/\delta v$, $\delta A/\delta \varphi$ when equated to zero yield equations (1), (2)? Here A is a functional defined by

$$(3) \quad A(u, v, \varphi, p) = \int_D L(u, v, \varphi, p, u_x, v_x, \varphi_x, p_x, u_y, v_y, \varphi_y, p_y, x, y) dx dy,$$

where L is a function (of the variables listed in parentheses in equation (3)) satisfying certain mathematical conditions regarding continuity, differentiability, integrability.

From the equation (3) we obtain following functional derivatives

$$(4) \quad \frac{\delta A}{\delta p} = \frac{\partial L}{\partial p} - \frac{\partial}{\partial x} \frac{\partial L}{\partial p_x} - \frac{\partial}{\partial y} \frac{\partial L}{\partial p_y},$$

$$(5) \quad \frac{\delta A}{\delta u} = \frac{\partial L}{\partial u} - \frac{\partial}{\partial x} \frac{\partial L}{\partial u_x} - \frac{\partial}{\partial y} \frac{\partial L}{\partial u_y},$$

$$(6) \quad \frac{\delta A}{\delta v} = \frac{\partial L}{\partial v} - \frac{\partial}{\partial x} \frac{\partial L}{\partial v_x} - \frac{\partial}{\partial y} \frac{\partial L}{\partial v_y},$$

$$(7) \quad \frac{\delta A}{\delta \varphi} = \frac{\partial L}{\partial \varphi} - \frac{\partial}{\partial x} \frac{\partial L}{\partial \varphi_x} - \frac{\partial}{\partial y} \frac{\partial L}{\partial \varphi_y},$$

provided that

$$\left(\frac{\partial L}{\partial p_x} \right)_B = \left(\frac{\partial L}{\partial p_y} \right)_B = 0.$$

Keeping in mind that L is a function of $u, v, \varphi, p, \dots$ we get from Eqs. (4)...(7)

$$\begin{aligned} \frac{\delta A}{\delta p} &= \frac{\partial L}{\partial p} - \frac{\partial^2 L}{\partial u \partial p_x} u_x - \frac{\partial^2 L}{\partial v \partial p_x} v_x - \frac{\partial^2 L}{\partial \varphi \partial p_x} \varphi_x - \frac{\partial^2 L}{\partial p \partial p_x} p_x - \frac{\partial^2 L}{\partial u \partial \varphi_y} u_y - \\ &\quad - \frac{\partial^2 L}{\partial v \partial p_y} v_y - \frac{\partial^2 L}{\partial \varphi \partial p_y} \varphi_y - \frac{\partial^2 L}{\partial p \partial p_y} p_y - \frac{\partial^2 L}{\partial u_x \partial p_x} u_{xx} - \frac{\partial^2 L}{\partial v_x \partial p_x} v_{xx} - \\ (8) \quad &\quad - \frac{\partial^2 L}{\partial \varphi_x \partial p_x} \varphi_{xx} - \left(\frac{\partial^2 L}{\partial u_x \partial p_y} + \frac{\partial^2 L}{\partial u_y \partial p_x} \right) u_{xy} - \left(\frac{\partial^2 L}{\partial v_x \partial p_y} + \frac{\partial^2 L}{\partial v_y \partial p_x} \right) v_{xy} - \\ &\quad - \left(\frac{\partial^2 L}{\partial \varphi_x \partial p_y} + \frac{\partial^2 L}{\partial \varphi_y \partial p_x} \right) \varphi_{xy} - 2 \frac{\partial^2 L}{\partial p_x \partial p_y} p_{xy} - \frac{\partial^2 L}{\partial p_x^2} p_{xx} - \frac{\partial^2 L}{\partial p_y^2} p_{yy} - \\ &\quad - \frac{\partial^2 L}{\partial u_y \partial p_y} u_{yy} - \frac{\partial^2 L}{\partial v_y \partial p_y} v_{yy} - \frac{\partial^2 L}{\partial \varphi_y \partial p_y} \varphi_{yy}, \\ \frac{\delta A}{\delta u} &= \frac{\partial L}{\partial u} - \frac{\partial^2 L}{\partial u \partial u_x} u_x - \frac{\partial^2 L}{\partial v \partial u_x} v_x - \frac{\partial^2 L}{\partial \varphi \partial u_x} \varphi_x - \frac{\partial^2 L}{\partial p \partial u_x} p_x - \frac{\partial^2 L}{\partial u \partial u_y} u_y - \\ (9) \quad &\quad - \frac{\partial^2 L}{\partial v \partial u_y} v_y - \frac{\partial^2 L}{\partial \varphi \partial u_y} \varphi_y - \frac{\partial^2 L}{\partial p \partial u_y} p_y - \frac{\partial^2 L}{\partial u_x^2} u_{xx} - \frac{\partial^2 L}{\partial v_x \partial u_x} v_{xx} - \\ &\quad - \frac{\partial^2 L}{\partial \varphi_x \partial u_x} \varphi_{xx} - \frac{\partial^2 L}{\partial p_x \partial u_x} p_{xx} - 2 \frac{\partial^2 L}{\partial u_x \partial u_y} u_{xy} - \frac{\partial^2 L}{\partial u_y^2} u_{yy} - \end{aligned}$$

$$\begin{aligned}
& - \left(\frac{\partial^2 L}{\partial v_x \partial u_y} + \frac{\partial^2 L}{\partial v_y \partial u_x} \right) v_{xy} - \left(\frac{\partial^2 L}{\partial \varphi_x \partial u_y} + \frac{\partial^2 L}{\partial \varphi_y \partial u_x} \right) \varphi_{xy} - \\
& - \left(\frac{\partial^2 L}{\partial p_x \partial u_y} + \frac{\partial^2 L}{\partial p_y \partial u_x} \right) p_{xy} - \frac{\partial^2 L}{\partial v_y \partial u_y} v_{yy} - \frac{\partial^2 L}{\partial \varphi_y \partial u_y} \varphi_{yy} - \frac{\partial^2 L}{\partial p_y \partial u_y} p_{yy},
\end{aligned}$$

$$\begin{aligned}
\frac{\delta A}{\delta v} &= \frac{\partial L}{\partial v} - \frac{\partial^2 L}{\partial u \partial v_x} u_x - \frac{\partial^2 L}{\partial v \partial v_x} v_x - \frac{\partial^2 L}{\partial \varphi \partial v_x} \varphi_x - \frac{\partial^2 L}{\partial p \partial v_x} p_x - \frac{\partial^2 L}{\partial u \partial v_y} u_y - \\
& - \frac{\partial^2 L}{\partial v \partial v_y} v_y - \frac{\partial^2 L}{\partial \varphi \partial v_y} \varphi_y - \frac{\partial^2 L}{\partial p \partial v_y} p_y - \frac{\partial^2 L}{\partial u_x \partial v_x} u_{xx} - \frac{\partial^2 L}{\partial v_x^2} v_{xx} - \\
(10) \quad & - \frac{\partial^2 L}{\partial \varphi_x \partial v_x} \varphi_{xx} - \frac{\partial^2 L}{\partial p_x \partial v_x} p_{xx} - \left(\frac{\partial^2 L}{\partial u_x \partial v_y} + \frac{\partial^2 L}{\partial u_y \partial v_x} \right) u_{xy} - \\
& - 2 \frac{\partial^2 L}{\partial v_x \partial v_y} v_{xy} - \left(\frac{\partial^2 L}{\partial \varphi_x \partial v_y} + \frac{\partial^2 L}{\partial \varphi_y \partial v_x} \right) \varphi_{xy} - \left(\frac{\partial^2 L}{\partial p_x \partial v_y} + \frac{\partial^2 L}{\partial p_y \partial v_x} \right) p_{xy} - \\
& - \frac{\partial^2 L}{\partial u_y \partial v_y} u_{yy} - \frac{\partial^2 L}{\partial v_y^2} v_{yy} - \frac{\partial^2 L}{\partial \varphi_y \partial v_y} \varphi_{yy} - \frac{\partial^2 L}{\partial p_y \partial v_y} p_{yy},
\end{aligned}$$

$$\begin{aligned}
\frac{\delta A}{\delta \varphi} &= \frac{\partial L}{\partial \varphi} - \frac{\partial^2 L}{\partial u \partial \varphi_x} u_x - \frac{\partial^2 L}{\partial v \partial \varphi_x} v_x - \frac{\partial^2 L}{\partial \varphi \partial \varphi_x} \varphi_x - \frac{\partial^2 L}{\partial p \partial \varphi_x} p_x - \\
& - \frac{\partial^2 L}{\partial u \partial \varphi_y} u_y - \frac{\partial^2 L}{\partial v \partial \varphi_y} v_y - \frac{\partial^2 L}{\partial \varphi \partial \varphi_y} \varphi_y - \frac{\partial^2 L}{\partial p \partial \varphi_y} p_y - \frac{\partial^2 L}{\partial \varphi_x^2} \varphi_{xx} - \\
(11) \quad & - \frac{\partial^2 L}{\partial u_x \partial \varphi_x} u_{xx} - \frac{\partial^2 L}{\partial v_x \partial \varphi_x} v_{xx} - \frac{\partial^2 L}{\partial p_x \partial \varphi_x} p_{xx} - \left(\frac{\partial^2 L}{\partial u_x \partial \varphi_y} + \frac{\partial^2 L}{\partial u_y \partial \varphi_x} \right) u_{xy} - \\
& - \left(\frac{\partial^2 L}{\partial v_x \partial \varphi_y} + \frac{\partial^2 L}{\partial v_y \partial \varphi_x} \right) v_{xy} - \left(\frac{\partial^2 L}{\partial p_x \partial \varphi_y} + \frac{\partial^2 L}{\partial p_y \partial \varphi_x} \right) p_{xy} - 2 \frac{\partial^2 L}{\partial \varphi_x \partial p_y} \varphi_{xy} - \\
& - \frac{\partial^2 L}{\partial \varphi_y^2} \varphi_{yy} - \frac{\partial^2 L}{\partial u_y \partial \varphi_y} u_{yy} - \frac{\partial^2 L}{\partial v_y \partial \varphi_y} v_{yy} - \frac{\partial^2 L}{\partial p_y \partial \varphi_y} p_{yy}.
\end{aligned}$$

If $\delta A/\delta p=0$, $\delta A/\delta u=0$, $\delta A/\delta v=0$ and $\delta A/\delta \varphi=0$ are to be equivalent to equations (1) and (2) then we must have

$$\begin{aligned}
\frac{\partial^2 L}{\partial u_x \partial p_x} &= \frac{\partial^2 L}{\partial v_x \partial p_x} = \frac{\partial^2 L}{\partial \varphi_x \partial p_x} = \frac{\partial^2 L}{\partial u_x \partial p_y} + \frac{\partial^2 L}{\partial u_y \partial p_x} = \frac{\partial^2 L}{\partial v_x \partial p_y} + \frac{\partial^2 L}{\partial v_y \partial p_x} = \\
(12) \quad &= \frac{\partial^2 L}{\partial \varphi_x \partial p_y} + \frac{\partial^2 L}{\partial \varphi_y \partial p_y} = \frac{\partial^2 L}{\partial p_x \partial p_y} = \frac{\partial^2 L}{\partial p_x^2} = \frac{\partial^2 L}{\partial p_y^2} = \frac{\partial^2 L}{\partial u_y \partial p_y} = \\
&= \frac{\partial^2 L}{\partial v_y \partial p_y} = \frac{\partial^2 L}{\partial \varphi_y \partial p_y} = 0,
\end{aligned}$$

$$\begin{aligned}
 (13) \quad \frac{\partial^2 L}{\partial v_x \partial u_x} &= \frac{\partial^2 L}{\partial \varphi_x \partial u_x} = \frac{\partial^2 L}{\partial u_x \partial u_y} = \frac{\partial^2 L}{\partial v_x \partial u_y} + \frac{\partial^2 L}{\partial v_y \partial u_x} = \frac{\partial^2 L}{\partial \varphi_x \partial u_y} + \frac{\partial^2 L}{\partial \varphi_y \partial u_x} = \\
 &= \frac{\partial^2 L}{\partial v_y \partial u_y} = \frac{\partial^2 L}{\partial \varphi_y \partial u_y} = \frac{\partial^2 L}{\partial \varphi_x \partial v_x} = \frac{\partial^2 L}{\partial v_x \partial v_y} = \frac{\partial^2 L}{\partial \varphi_x \partial v_y} + \frac{\partial^2 L}{\partial \varphi_y \partial v_x} = \\
 &= \frac{\partial^2 L}{\partial \varphi_y \partial v_y} = \frac{\partial^2 L}{\partial \varphi_x \partial \varphi_y} = 0,
 \end{aligned}$$

$$(14) \quad \frac{\partial^2 L}{\partial v_x^2} = \frac{\partial^2 L}{\partial v_y^2} = \frac{\partial^2 L}{\partial u_x^2} = \frac{\partial^2 L}{\partial u_y^2} = \nu + \kappa, \quad \frac{\partial^2 L}{\partial \varphi_x^2} = \frac{\partial^2 L}{\partial \varphi_y^2} = \gamma.$$

It may be noted the validity of Eqs. (12), (13) and (14) is not a sufficient condition but only a necessary one for the desired equivalence.

From Eqs. (12), (13) and (14) we get

$$\begin{aligned}
 (15) \quad L &= E + E_1 u_x + E_2 v_x + E_3 \varphi_x + E_4 u_y + E_5 v_y + E_6 \varphi_y + E_7 p_x + E_8 p_y + \\
 &+ \frac{1}{2}(\nu + \kappa)[u_x^2 + v_x^2 + u_y^2 + v_y^2] + \frac{1}{2}\gamma(\varphi_x^2 + \varphi_y^2) + E_9 p_x u_y + E_{10} p_x v_y + \\
 &+ E_{11} p_x \varphi_y + E_{12} p_y u_x + E_{13} p_y v_x + E_{14} p_y \varphi_x + E_{15} u_x v_y + E_{16} u_x \varphi_y + \\
 &+ E_{17} v_x u_y + E_{18} v_x \varphi_y + E_{19} \varphi_x u_y + E_{20} \varphi_x v_y,
 \end{aligned}$$

and

$$(16) \quad E_{11} + E_{14} = E_{15} + E_{17} = E_{16} + E_{19} = E_{18} + E_{20},$$

where $E, E_1, E_2, \dots, E_{20}$ are arbitrary functions of u, v, φ, p and their dependence on these variables is not exhibited.

Now the expressions for $\delta A / \delta p, \delta A / \delta u, \delta A / \delta v, \delta A / \delta \varphi$ given by (8)...(11) reduce to

$$\begin{aligned}
 (17) \quad \frac{\delta A}{\delta p} &= \frac{\partial L}{\partial p} - \frac{\partial^2 L}{\partial u \partial p_x} u_x - \frac{\partial^2 L}{\partial v \partial p_x} v_x - \frac{\partial^2 L}{\partial \varphi \partial p_x} \varphi_x - \frac{\partial^2 L}{\partial p \partial p_x} p_x - \\
 &- \frac{\partial^2 L}{\partial u \partial p_y} u_y - \frac{\partial^2 L}{\partial v \partial p_y} v_y - \frac{\partial^2 L}{\partial \varphi \partial p_y} \varphi_y - \frac{\partial^2 L}{\partial p \partial p_y} p_y,
 \end{aligned}$$

$$\begin{aligned}
 (18) \quad \frac{\delta A}{\delta u} &= \frac{\partial L}{\partial u} - \frac{\partial^2 L}{\partial u \partial u_x} u_x - \frac{\partial^2 L}{\partial v \partial u_x} v_x - \frac{\partial^2 L}{\partial \varphi \partial u_x} \varphi_x - \frac{\partial^2 L}{\partial p \partial u_x} p_x - \frac{\partial^2 L}{\partial u \partial u_y} u_y - \\
 &- \frac{\partial^2 L}{\partial v \partial u_y} v_y - \frac{\partial^2 L}{\partial \varphi \partial u_y} \varphi_y - \frac{\partial^2 L}{\partial p \partial u_y} p_y - (\nu + \kappa)(u_{xx} + u_{yy}),
 \end{aligned}$$

$$\begin{aligned}
 (19) \quad \frac{\delta A}{\delta v} &= \frac{\partial L}{\partial v} - \frac{\partial^2 L}{\partial u \partial v_x} u_x - \frac{\partial^2 L}{\partial v \partial v_x} v_x - \frac{\partial^2 L}{\partial \varphi \partial v_x} \varphi_x - \frac{\partial^2 L}{\partial p \partial v_x} p_x - \frac{\partial^2 L}{\partial u \partial v_y} u_y - \\
 &- \frac{\partial^2 L}{\partial v \partial v_y} v_y - \frac{\partial^2 L}{\partial \varphi \partial v_y} \varphi_y - \frac{\partial^2 L}{\partial p \partial v_y} p_y - (\nu + \kappa)(v_{xx} + v_{yy}),
 \end{aligned}$$

$$(20) \quad \frac{\delta A}{\delta \varphi} = \frac{\partial L}{\partial \varphi} - \frac{\partial^2 L}{\partial u \partial \varphi_x} u_x - \frac{\partial^2 L}{\partial v \partial \varphi_x} v_x - \frac{\partial^2 L}{\partial \varphi \partial \varphi_x} \varphi_x - \frac{\partial^2 L}{\partial p \partial \varphi_x} p_x - \frac{\partial^2 L}{\partial u \partial \varphi_y} u_y - \\ - \frac{\partial^2 L}{\partial v \partial \varphi_y} v_y - \frac{\partial^2 L}{\partial \varphi \partial \varphi_y} \varphi_y - \frac{\partial^2 L}{\partial p \partial \varphi_y} p_y - \gamma(\varphi_{xx} + \varphi_{yy}).$$

It is obvious that the terms in Eq. (15) containing $E, E_7 \dots E_{20}$ cannot reproduce the inertia terms. We choose

$$(21) \quad E_7 = \frac{u}{\rho}, \quad E_8 = \frac{v}{\rho}, \quad E = E_9 = E_{10} = E_{11} = E_{12} = E_{13} = E_{15} = E_{16} = E_{18} = 0,$$

which means

$$(22) \quad L = E_1 u_x + E_2 v_x + E_3 \varphi_x + E_4 u_y + E_5 v_y + E_6 \varphi_y + \frac{u}{\rho} p_x + \frac{v}{\rho} p_y + \\ + \frac{1}{2}(\nu + \kappa)[u_x^2 + v_x^2 + u_y^2 + v_y^2] + \frac{1}{2}\gamma(\varphi_x^2 + \varphi_y^2).$$

Substituting for L from (22) into (17)–(20), we get

$$(23) \quad \frac{\delta A}{\delta p} = \left(\frac{\partial E_1}{\partial p} - \frac{1}{\rho} \right) u_x + \left(\frac{\partial E_5}{\partial p} - \frac{1}{\rho} \right) v_y + \frac{\partial E_2}{\partial p} v_x + \frac{\partial E_3}{\partial p} \varphi_x + \frac{\partial E_4}{\partial p} u_y + \frac{\partial E_6}{\partial p} \varphi_y,$$

$$(24) \quad \frac{\delta A}{\delta u} = \left(\frac{\partial E_2}{\partial u} - \frac{\delta E_1}{\delta v} \right) v_x + \left(\frac{\partial E_3}{\partial u} - \frac{\partial E_1}{\partial \varphi} \right) \varphi_x + \left(\frac{\partial E_5}{\partial u} - \frac{\partial E_4}{\partial v} \right) v_y - \frac{\partial E_4}{\partial p} p_y + \\ + \left(\frac{\partial E_6}{\partial u} - \frac{\partial E_4}{\partial \varphi} \right) \varphi_y + \left(\frac{1}{\rho} - \frac{\partial E_1}{\partial p} \right) p_x - (\nu + \kappa)(u_{xx} + u_{yy}),$$

$$(25) \quad \frac{\delta A}{\delta v} = \left(\frac{\partial E_1}{\partial v} - \frac{\partial E_2}{\partial u} \right) u_x + \left(\frac{\partial E_4}{\partial v} - \frac{\partial E_5}{\partial u} \right) u_y + \left(\frac{\partial E_3}{\partial v} - \frac{\partial E_2}{\partial \varphi} \right) \varphi_x - \frac{\partial E_2}{\partial p} p_x + \\ + \left(\frac{\partial E_6}{\partial v} - \frac{\partial E_5}{\partial \varphi} \right) \varphi_y + \left(\frac{1}{\rho} - \frac{\partial E_5}{\partial p} \right) p_y - (\nu + \kappa)(v_{xx} + v_{yy}),$$

$$(26) \quad \frac{\delta A}{\delta \varphi} = \left(\frac{\partial E_1}{\partial \varphi} - \frac{\partial E_3}{\partial u} \right) u_x + \left(\frac{\partial E_2}{\partial \varphi} - \frac{\partial E_3}{\partial v} \right) v_x + \left(\frac{\partial E_4}{\partial \varphi} - \frac{\partial E_6}{\partial u} \right) u_y + \\ + \left(\frac{\partial E_5}{\partial \varphi} - \frac{\partial E_6}{\partial v} \right) v_y - \frac{\partial E_3}{\partial p} p_x - \frac{\partial E_6}{\partial p} p_y - \gamma(\varphi_{xx} + \varphi_{yy}).$$

From these expressions it is clear that no functions E_1, E_2, E_3, E_4, E_5 , and E_6 exist which will reproduce the inertia terms in Eqs. (1) and (2). We therefore conclude that there does not exist a single functional $A[u, v, \varphi, p]$ whose functional derivatives with respect to u, v and φ when equated to zero would yield Eqs. (1) and (2).

As in [1] it can be established a connection between the two-dimensional flow equations of a micropolar fluid and certain functionals. To establish this, let

$$(27) \quad D(u, v, \varphi, p) = \iint_D \left[\frac{v+\kappa}{2} (\operatorname{div} u)^2 + \frac{v+\kappa}{2} (\operatorname{rot} u)^2 + \kappa(v\varphi_x - u\varphi_y) + \right. \\ \left. + \frac{\gamma}{2} (\operatorname{grad} \varphi)^2 - \kappa\varphi^2 + \frac{1}{\rho} u \operatorname{grad} p \right] dx dy,$$

and

$$(28) \quad E(u, v) = \iint_D E_1 u_x + E_2 v_x + E_3 u_y + E_4 v_y dx dy, \\ F(u, v, \varphi) = \iint_D [F_1 \varphi_x + F_2 \varphi_y] dx dy,$$

where E_i , ($i=1, 2, 3, 4$) are functions of u, v and F_1, F_2 are functions of u, v, φ .

We then get

$$(29) \quad \frac{\delta D}{\delta u} = \frac{1}{\rho} p_x - (v+\kappa)(u_{xx} + u_{yy}) - \kappa\varphi_y, \\ \frac{\delta D}{\delta v} = \frac{1}{\rho} p_y - (v+\kappa)(v_{xx} + v_{yy}) + \kappa\varphi_x, \\ \frac{\delta D}{\delta \varphi} = -\gamma(\varphi_{xx} + \varphi_{yy}) - \kappa(v_x - u_y) + 2\kappa\varphi, \\ \frac{\delta D}{\delta p} = -\frac{1}{\rho}(u_x + v_y),$$

and

$$(30) \quad \frac{\delta E}{\delta v} = -(u u_x + v u_y), \quad \frac{\delta E}{\delta u} = u v_x + v v_y, \\ \frac{\delta F}{\delta v} - \frac{\delta F}{\delta u} = u \varphi_x + v \varphi_y,$$

if we choose E, F_j ($i=1, 2, 3, 4; j=1, 2$) so as to satisfy

$$(31) \quad \frac{\partial E_2}{\partial u} - \frac{\partial E_1}{\partial v} = u, \quad \frac{\partial E_4}{\partial u} - \frac{\partial E_3}{\partial v} = v, \\ \frac{\partial F_1}{\partial v} - \frac{\partial F_2}{\partial u} = u, \quad \frac{\partial F_2}{\partial v} - \frac{\partial F_1}{\partial u} = v.$$

Using Eqs. (29) and (30) we can write Eqs. (1) and (2) equivalently as

$$(32) \quad \frac{\delta D}{\delta p} = 0,$$

$$(33) \quad \frac{\delta E}{\delta v} = \frac{\delta D}{\delta u}, \quad \frac{\delta E}{\delta u} = -\frac{\delta D}{\delta v}, \quad a^2 \left[\frac{\delta F}{\delta u} - \frac{\delta F}{\delta v} \right] = \frac{\delta D}{\delta \varphi}.$$

In order to establish the existence of E and F compatible with Eqs. (31), we observe that among infinitely many solutions of (31) a simple solution is

$$E_1 = E_4 = 0, \quad E_2 = \frac{1}{2}u^2, \quad E_3 = -\frac{1}{2}v^2,$$

$$F_1 = u + v - \frac{1}{2}u^2, \quad F_2 = u + v + \frac{1}{2}v^2,$$

giving

$$E = \iint_D \frac{1}{2} (u^2 v_x - v^2 u_y) dx dy,$$

$$F = \iint_D \left[(u+v)(\varphi_x + \varphi_y) + \frac{1}{2} (v^2 \varphi_y - u^2 \varphi_x) \right] dx dy.$$

Remark. If one takes two functionals only, D in (27) and E of the form

$$E(u, v, \varphi) = \iint_D [E_1 u_x + E_2 v_x + E_3 \varphi_x + E_4 u_y + E_5 v_y + E_6 \varphi_y] dx dy,$$

where E_i , ($i=1, 2, \dots, 6$) are functions of u, v, φ then we can easily observe that no functions $E_1, E_2, \dots, E_6$ exist so as the third equation of (33) holds.

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O FORMULARE VARIATIONALĂ A ECUAȚIILOR CURGERII FLUIDELOR MICROPOLARE

(Rezumat)

Se arată că ecuațiile curgerii bidimensionale a unui fluid micropolar incompresibil sînt echivalente cu un sistem de ecuații funcționale.

In order to establish the existence of E and K compatible with F and (31), we observe that among infinitely many solutions of (31) a simple solution is

$$E_1 = E_2 = 0, \quad E_3 = \frac{1}{2}u^2, \quad E_4 = \frac{1}{2}v^2, \\ E_5 = u + v - \frac{1}{2}u^2, \quad E_6 = u + v + \frac{1}{2}v^2,$$

giving

$$E = \iint_D \frac{1}{2} (u^2 + v^2 - uv) dx dy, \\ F = \iint_D \left[(u+v)(\varphi_1 + \varphi_2) + \frac{1}{2} (u^2 \varphi_1 - v^2 \varphi_2) \right] dx dy.$$

Remark. It contains two functionals only, D in (27) and E of the form

$$E(u, v, \varphi) = \iint_D [E_1 u + E_2 v + E_3 \varphi_1 + E_4 \varphi_2 + E_5 u + E_6 v + E_7 \varphi_1] dx dy,$$

where $E_1, \dots, E_7$ are functions of u, v, φ then we can easily observe that no functions $F_1, F_2, \dots, F_7$ exist so as the third equation of (31) holds.

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О ФОРМУЛАХ ВАРИАЦИОНАЛЬНОГО УРАВНЕНИЯ ЛИНДЕЛЮН-МИКРОПОЛЯРА

(Известия)

Мы ставим задачу вывести функционалы E и F для микрополя инвариантного относительно системы деформаций.

ON REDUCTION OF THE REPRESENTATIONS FROM d_2 -SERIES OF $SU(2,2)$ TO THE MAXIMAL COMPACT SUBGROUP

BY

GH. ZET

1. Introduction. The unitary irreducible representations of the non-compact groups are infinite dimensional. When these representations are restricted to a compact subgroup they become reducible. Therefore, we can write such a representation as an infinite direct sum of unitary irreducible representations of the compact subgroup [1].

The reduction of the representations of a group to one of its subgroup is important, for example, when the representation theory of noncompact groups is applied to the elementary particles. For such a purpose, it is necessary to solve the following problems:

- i) to give methods for construction of unitary irreducible representations of groups in Hilbert spaces with discrete basis;
- ii) to know what is the reduction of an irreducible representation when restricted to a compact subgroup.

For certain classes of groups, e.g. classical complex groups, it is possible to construct irreducible unitary representations using the induced representation method. For applications in elementary particles, it is necessary to have the corresponding representations in a discrete basis (whose elements can be associated with particle states) and their reduction to the maximal compact subgroup [2]. In this work we will show what is the reduction of the representations from continuous nondegenerate d_2 -series of $SU(2,2)$ to the maximal compact subgroup $K_m = SU(2) \times SU(2) \times U(1)$. Having this reduction solved, we can then construct the representations from d_2 -series in a K_m -basis [3]. This construction will be taken up elsewhere.

2. The Continuous Nondegenerate d_2 -series. The $SU(2,2)$ group is defined as the set of all transformations of a 4-dimensional linear complex space that leave invariant the following quadratic form:

$$(1) \quad |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2.$$

Equivalent, the $SU(2,2)$ group can be defined as the set of all matrix g complex, unimodular, 4×4 -dimensional, that have the property

$$(2) \quad g^+ S g = S, \quad S = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The sign „+“ notes the Hermitian conjugation of a matrix.

To describe the representations from d_2 -series, we will introduce first of all the following two subgroups of $SU(2,2)$:

$$(3) \quad \Lambda \ni \Lambda = \begin{pmatrix} \bar{\lambda}^{-1} & -\mu/\Delta & \cdot & \lambda_1 \\ 0 & \bar{\lambda}/\Delta & \cdot & \cdot \\ 0 & 0 & \lambda^{-1}\Delta & \mu \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad Y \ni y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\bar{\omega} & 1 & 0 & 0 \\ a & b & 1 & 0 \\ \omega a + c & \omega b - \bar{a} & \omega & 1 \end{pmatrix}.$$

where λ_1 is a 2×2 matrix and a, b, c, ω are arbitrary complex numbers. It is easy to verify that these matrix are unimodular and satisfy the condition (2), i.e. they are elements of $SU(2,2)$.

Every element $g \in SU(2,2)$ can be written in a unique way in the form [4]

$$(4) \quad g = \Lambda y, \quad \Lambda \in \Lambda, \quad y \in Y.$$

In particular, if $g \in SU(2,2)$ and $y \in Y$, then we can write

$$(5) \quad gy = \Lambda y', \quad y' \in Y, \quad \Lambda \in \Lambda.$$

The representations $g \rightarrow R(g)$ from d_2 -series of $SU(2,2)$ are constructed in the Hilbert space H of all functions $f(y)$, defined over Y subgroup, which are square-integrable. The action of the unitary operator $R(g)$ on a such function $f(y)$ is given by formula

$$(6) \quad [R(g)f](y) = \mu(g, y)f(y'), \quad f \in H,$$

where

$$(7) \quad \mu(g, y) = |\Delta|^{i\rho_1-1} (\text{sign } \Delta)^\varepsilon |\lambda|^{m-i\rho_1+i\rho_2-2} \lambda^{-m}$$

and g, y, y' satisfy the condition (5). Here ρ_1 and ρ_2 are arbitrary real numbers, m is an arbitrary integer number, and ε is a sign parameter. The representations so constructed are noted by $(m, \rho_1, \rho_2, \varepsilon)$ and they constitute the d_2 -series; there is a representation for every set of values of the parameters m, ρ_1, ρ_2 and ε .

3. The Reduction of d_2 -series to the K_m Subgroup. When we restrict the representations from d_2 -series to the maximal compact subgroup K_m , i.e. when $z \in K_m$, they depend on the continuous parameters ρ_1 and ρ_2 . Or, it is known that unitary irreducible representations of a compact group are characterized by discrete indices only. The situation changes if we consider an equivalent representation $R'(g) = SR(g)S^{-1}$, where S is a transformation in H defined as:

$$(8) \quad (Sf)(y) = \exp [i\Phi(y)]f(y),$$

$$(9) \quad \Phi(y) = -\frac{1}{2} \rho_1 \ln (1 + 2|a|^2 + |a|^4 + |b|^2 + |c|^2 + |b|^2|c|^2 + 2|b||c||a|^2) + \\ + \frac{1}{2} (\rho_1 - \rho_2) \ln (1 + |\omega|^2 + |\omega b - \bar{a}|^2 + |\omega a + c|^2).$$

Then, Eq. (6) can be rewritten as

$$(10) \quad [R'(g)f](y) = \exp [i\Phi(y) - i\Phi(y')] \mu(g, y) f(y').$$

It is easy to verify that for $g \in K_m$, i.e. $gg^+ = I$, the right-hand side of equation (10) has not a dependence on ρ_1 and ρ_2 parameters. Correspondently, the infinitesimal generators of K_m do not depend on ρ_1 and ρ_2 .

Then, it is possible to construct the representations of $SU(2,2)$ from d_2 -series in a K_m basis, in which the problem of reduction to the subgroup K_m is solved. This will be taken up elsewhere.

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ASUPRA REDUCERII REPREZENTĂRIILOR DE LA SERIILE d_2 ALE LUI $SU(2,2)$ LA SUBGRUPUL COMPACT MAXIMAL

(Rezumat)

Reprezentările ireductibile unitare ale grupurilor necompacte sînt infinit dimensionale. Pentru anumite clase de grupuri, de pildă grupurile complexe clasice, este posibil de construit reprezentări unitare ireductibile folosind metoda reprezentării induse. În vederea aplicațiilor la particulele elementare este necesar de exprimat respectiva reprezentare într-o bază discretă (ale cărei elemente se pot asocia cu stările particulelor).

În Nota prezentă se studiază reducerea reprezentărilor seriilor d_2 nedegenerate continue ale lui $SU(2,2)$ la subgrupul compact maximal $K_m = SU(2) \times SU(2) \times U(1)$, ceea ce permite construirea reprezentărilor seriilor d_2 în baza K_m .

Then, Eq. (6) can be rewritten as

$$(10) \quad [R(x)(K) - \exp(10^x) - 10^x] / (x^2 - 1) = 0.$$

It is easy to verify that for $x \in K$, i.e. $x \in \mathbb{R}$, the right-hand side of equation (10) has not a dependence on x and x parameters. Correspondingly, the infinitesimal generators of K do not depend on x and x . Then, it is possible to construct the representations of $SL(2, \mathbb{R})$ from x -series in a K -basis, in which the problem of reduction to the subgroup K is solved. This will be taken up elsewhere.

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ASUPRA REDUCERII REPREZENTĂRII DE LA SERIEA A_1 ALIE LUI $SL(2, \mathbb{R})$ LA SUBGRUPUL COMPACT MAXIMAL

(Rezumat)

Reprezentările infinit-dimensionale unitare ale grupului compact maximal al lui $SL(2, \mathbb{R})$ sunt studiate în acest articol. În primul rând se arată că aceste reprezentări sunt reducibile la reprezentările de seriea A_1 . Apoi se arată că aceste reprezentări sunt reducibile la reprezentările de seriea A_1 și la reprezentările de seriea A_1 .

La sfârșitul acestui articol se arată că aceste reprezentări sunt reducibile la reprezentările de seriea A_1 și la reprezentările de seriea A_1 .

A NEWTONIAN PHYSICAL SIGNIFICANCE OF REISSNER-NORDSTRÖM SOLUTION

BY

CORNELIU D. CIUBOTARIU

The starting point for this work is the Reissner-Nordström solution of the Einstein equations for a particle of mass M and charge Q :

$$(1) \quad ds^2 = A^{-1} dr^2 - A c^2 dt^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2),$$

where

$$(2) \quad A(r) = 1 - \frac{a}{r} + \frac{b}{r^2}, \quad a = \frac{2GM}{c^2}, \quad b = \frac{GQ^2}{4\pi\epsilon_0 c^4}$$

with G the Newtonian gravitational constant.

In order to obtain some physical significances of the Reissner-Nordström metric we must introduce a standard of reference. The point is that an observer measures not with respect to some global coordinate system, but with respect to some suitable locally-chosen tetrad which is determined by local observations, such that any fundamental observer using such a reference frame would record the same observations.

To find the acceleration field by a charged mass we will use the frame vectors along the coordinate axes numbered $(0, 1, 2, 3)$ in the order (ct, r, θ, φ) , that is ([8], Eq. (2.6))

$$(3) \quad \lambda_{(\alpha)}^\mu = \delta_\alpha^0 \delta_0^\mu (-g_{00})^{-\frac{1}{2}} + \delta_\alpha^1 \delta_1^\mu (g_{11})^{-\frac{1}{2}} + \delta_\alpha^2 \delta_2^\mu r^{-1} + \delta_\alpha^3 \delta_3^\mu r^{-1} \csc \theta.$$

This system of reference tetrads is defined in a unique manner because it describes the world line of a stationary test particle in the central (Reissner-Nordström) field of a mass M and charge Q .

Cattaneo [3] has defined a physical system of reference as a fluid of reference, that is ∞^3 ideal particle whose time tracks are the coordinate lines $x^0 = \text{var}$. The congruence of this coordinate lines determines a unique field $u^\mu = \lambda_{(0)}^\mu$ of unit tangents. In any point of the space-time this vector may be identified with the 4-velocity of the stationary test particle in the gravitational field determined by $g_{\alpha\beta}$. From the invariant Frenet-Serret formulae ([11], p. 10) we have obtained exactly the frame vectors (3). Furthermore, these vectors are easily shown to satisfy the Fermi-

Walker transport and, therefore, they constitute a frame of reference which must appear to give the correct relativistic generalization of the Newtonian concept.

In the Reissner-Nordström field, the 4-acceleration vector of the supported observer is

$$(4) \quad \frac{\delta \lambda_{(0)}^{\nu}}{\delta s} = \Gamma_{00}^{\nu} u^0 u^0 = a^{\nu} = (0, -\frac{1}{2} g_{00,1}, 0, 0) = \left(0, \frac{a}{2r^2} - \frac{b}{r^3}, 0, 0\right),$$

and

$$(5) \quad \frac{\alpha}{c^2} = (a^{\nu} a_{\nu})^{\frac{1}{2}} = \left(\frac{a}{2r^2} - \frac{b}{r^3}\right) \left(1 - \frac{a}{r} - \frac{b}{r^2}\right)^{-\frac{1}{2}}$$

is the first curvature of the world line of this observer. Evidently α equals the 3-force per unit mass experienced by a supported test particle (for the analogous Schwarzschild field see [9], p. 2085 and [6] p. 383). For easy reference we give the corresponding expression of α in the Schwarzschild field:

$$(6) \quad \frac{\alpha_S}{c^2} = \frac{a}{2r^2} \left(1 - \frac{a}{r}\right)^{-\frac{1}{2}}.$$

Now we can draw some conclusions:

1. If we consider the gravitational field of an electron, at the classical radius of the electron ($r_e = 2b/a \sim 10^{-13}$ cm, see [1], p. 398) the effective gravitational force is zero ($\alpha = 0$). We have thus obtained, by another method, a similar result to that of Ross [10] and Basano [2]. If $r < r_e = 2b/a$, the gravitational force (5) becomes negative, that is, the electric charges generates antigravity. We note that the classical electric radius r_e is much greater than the corresponding gravitational radius (Schwarzschild radius) r_s :

$$r_e \gg r_s = a = \frac{2GM}{c^2} \sim 10^{-55} \text{ cm}.$$

These results are not intrinsic and thus the antigravitation, as well as the gravitation, depends on the reference frame.

2. At the present time the universe is observed to be expanding. If we apply the above results to the dimensions of the universe:

$$\frac{r_e}{r_s} = \frac{1}{8\pi\epsilon_0 G} \frac{Q^2}{M^2} \sim 10^{20} \frac{Q^2}{M^2},$$

it is possible that $r_e > r_s$ and therefore, at large distances when the spherical symmetry is realized, the repulsive field is predominantly. Such hypotheses as „we may think of the universe itself as a black hole“ are known (see [5] and [4]).

3. As a final (but by no means less important) remark we want to point out that the consistency of the (electric) superposition principle is difficult to interpret. From an electric point of view a static spherical symmetric distribution of two charges ($+Q$, $-Q$) with the same center does not manifest electric properties outside the distribution, but from an inertial

point of view the proper gravitational mass of the two corresponding electric fields remains. We suppose that it exists an „inertial superposition principle”: The metric (1) is independent of the sign of charge Q and thus for a spherical symmetric distribution of two charges $(+Q, -Q)$ the corresponding metric may be obtained from Eq. (1) if we make the substitution $M \rightarrow 2M$, and $Q \rightarrow 2Q$, the gravitational field created by the charge $+Q$ being identical with that generated by the charge $-Q$. For example, the gravitational field of a neutron must be described by a metric of the type (1), because the neutron contains electric charges.

We note that the idea of the antigravity related to the expansion of the universe appears in a recent paper by Molteni and Ziino [7].

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O SEMNIFICAȚIE FIZICĂ NEWTONIANĂ A SOLUȚIEI REISSNER-NORD STRÖM

(Rezumat)

Cu ajutorul tehnicii tetradelor sînt cercetate unele sensuri fizice ale metricii Reissner-Nordström. Se arată că masa gravitațională proprie a cîmpului electric datorat unei sarcini electrice cu simetrie sferică conduce la o atenuare a forței gravitaționale. Din considerații relativist generale, se propune formularea unui nou principiu de superpoziție (inertial). Sînt indicate unele aplicații posibile în domeniul particulelor elementare și al expansiunii Universului.

point of view the proper gravitational mass of the two corresponding electric fields remains. We suppose that it exists an "inertial superposition principle": The metric (1) is independent of the sign of charge Q and thus for a spherical symmetric distribution of two charges $+Q$ and $-Q$ the corresponding metric may be obtained from Eq. (1) if we make the substitution $W \rightarrow 2W$, and $Q \rightarrow 2Q$, the gravitational field created by the charge $+Q$ being identical with that generated by the charge $-Q$. For example, the gravitational field of a neutron must be described by a metric of the type (1), because the neutron contains electric charges.

We note that the idea of the superativity related to the expansion of the universe appears in a recent paper by M. J. L. and Z. J. [7].

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O SCHWACHFELD THEORIE EINER NEUTRONEN-STRÖMUNG

(Russian)

Ein zentraler, ruhender, elektrisch-neutraler, kugelförmiger Körper mit elektrischer Ladung Q erzeugt ein elektrostatisches Feld. Ein zentraler, ruhender, elektrisch-neutraler, kugelförmiger Körper mit elektrischer Ladung Q erzeugt ein elektrostatisches Feld. Ein zentraler, ruhender, elektrisch-neutraler, kugelförmiger Körper mit elektrischer Ladung Q erzeugt ein elektrostatisches Feld. Ein zentraler, ruhender, elektrisch-neutraler, kugelförmiger Körper mit elektrischer Ladung Q erzeugt ein elektrostatisches Feld.

INVESTIGATIONS OF THE THERMAL SHRINKAGE**ESTIMATION METHOD OF THE INTERNAL STRESSES ACCUMULATED
IN POLY (ETHYLENE TEREPHTHALATE) YARNS DURING
THE TECHNOLOGICAL PROCESSES**

BY

SOFIA MELINTE, M. LEANCĂ and ALEXANDRINA JEFLEA

1. Introduction. During the technological processes of fibre formation, like spinning, stretching, orientation etc, important strains appear, resulting in the accumulation of the internal stresses, which will constitute elements of instability in the fibre structure and properties. The melt-spinning and cooling the polymer, its subsequent stretching a.s.o. will lead to a thermodynamic system which, in time, will tend to the equilibrium.

This evolution can be limited in time by the dimensional stabilization thermal treatments which, in turn, will produce the relaxation of the internal stresses, accompanied by important modifications of the sample dimensions and structure (secondary recrystallization of the amorphous regions, molecular disorientation etc). The control of the fibre thermal shrinkage can be an important method for estimating the internal stresses accumulated in the sample during a technological process, that can be a precious instrument available for every manufacturer interested in the thermal treatment efficiency.

In the control laboratories, the shrinkage determination can become a routine method that can be used in controlling the technological processes, so that to obtain a yarn with an exactly known stress which can be modified if necessary.

The rate of the dimensional modifications produced during the different processes can be controlled by polymer viscoelastic characteristics. On the basis of viscoelastic strain theory, dimensional modifications produced by thermal treatment can be explained. To make evident the results obtained in this work, concerning the thermal shrinkage, we shall briefly present the viscoelastic strain theory in polymers, on whose bases the dimensional variations of fibres during the thermal treatment can be explained.

2. Theoretical Considerations. Based on the Kelvin-Voight rheological model, the authors have established a correlation between the internal stresses appeared during the technological formation processes of yarns and their shrinkage at different temperatures. The results show that in the absence of an external stress on the fibre during the annealing treatment, the shrinkage is always positive, having the same value as its viscoelastic component. On the basis

of the stress-strain birefringence relations obtained from the kinetic theory of rubber elasticity, a stress-optical coefficient was established for the PET fibres.

The synthetic fibres are viscoelastic materials in which the stress and strain are time dependent [1]. With the view of determining the elongation in the case of a simple model of viscoelastic material, we shall consider the Kelvin-Voight rheological model (see Fig. 1) consisting of a series of viscoelastic elements with elastic modulus

E_i and elongation viscosity η_i^* . If we denote by a_i the fraction of elements with the elasticity modulus E_i , viscosity η_i^* , elongation ε and tensile stress σ , then we can write the equation system

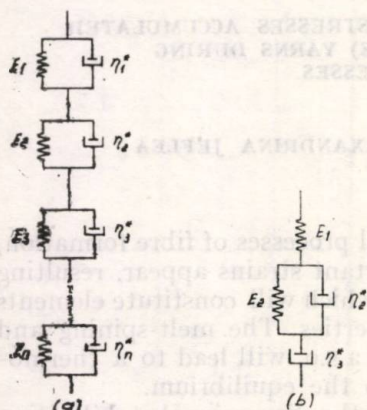


Fig. 1 — Rheological model for viscoelastic bodies: *a* — the general case; *b* — the simplified model, consisting of three relaxation times: $\tau_1=0$; $\tau_2=\eta_2^*/E_2$ and $\tau_3=0$.

$$\tau_3=0, \quad \varepsilon = \sum_{i=1}^n a_i \varepsilon_i,$$

(1)

$$\varepsilon_i + \tau_i \dot{\varepsilon}_i = \frac{\sigma}{E_i}, \quad \sum_{i=1}^n a_i = 1,$$

where $\tau_i = \eta_i^*/E_i$ is the relaxation time. The contribution of the elements with different relaxation times $a_i(\tau_i)$ making the so-called „relaxation time spectrum“ is a characteristic of material, determining the rate with which the strain and stress are changing in time.

For a quantitative analysis of some viscoelastic models, the model (a) in Fig. 1 can be reduced to a simplified model (b), consisting of only three fundamental strain elements: the perfectly elastic component ε_1 , to whom a relaxation time $\tau_1=0$ is corresponding; a viscoelastic component ε_2 , with a finite relaxation time $\tau_2=\eta_2^*/E_2$, and a viscous one ε_3 , with a relaxation time $\tau_3=\infty$. These three components contributing to the statistic elongation are given by the equations

$$(2) \quad \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon, \quad \varepsilon_1 = \frac{\sigma}{E_1}, \quad \varepsilon_2 + \tau_2 \dot{\varepsilon}_2 = \frac{\sigma}{E_2}, \quad \dot{\varepsilon}_3 = \frac{\sigma}{\eta_3^*}.$$

By combining the equations of system (2), a relation results between the macroscopic elongation and the applied stress σ , namely

$$(3) \quad \tau_2 \ddot{\varepsilon} + \dot{\varepsilon} = \frac{\tau_2}{E_1} \ddot{\sigma} + \left(\frac{1}{E_1} + \frac{1}{E_2} + \frac{\tau_2}{\eta_2^*} \right) \dot{\sigma} + \frac{\sigma}{\eta_3^*},$$

where $\varepsilon, \dot{\varepsilon}, \ddot{\varepsilon}$ and $\sigma, \dot{\sigma}, \ddot{\sigma}$ represent the first and the second time derivatives of the elongation and the stress respectively.

The equation (3) can be solved in different particular conditions [1].

In the case of the stretched fibres subjected to an isothermal deformation at a constant tensile stress σ , the elongation at $t=0$ will be

$$(4) \quad \varepsilon(t=0) = \varepsilon_{20} + \varepsilon_{30},$$

where ε_{20} is the viscoelastic component at $t=0$, which is proportional to the internal stress of the stretched fibre and determines the tensile force for viscoelastic recovery, or the relaxation stress, and ε_{30} is the irreversible (viscous) component.

By imposing the initial condition for the deformations during the thermal treatment of the fibres, i.e.

$$(5) \quad \dot{\varepsilon}(t=0) = \sigma \left(\frac{1}{\eta_2^*} + \frac{1}{\eta_3^*} \right) - \varepsilon_{20} \frac{E_2}{\eta_2^*}$$

and with the condition $\ddot{\sigma} = \ddot{\sigma} = 0$, the general equation (3) becomes

$$(6) \quad \varepsilon(t) = \varepsilon_{30} + \frac{\sigma t}{\eta_3^*} + \frac{\sigma}{E_2} \left[1 - \exp\left(-\frac{t}{\tau_2}\right) + \varepsilon_{20} \exp\left(-\frac{t}{\tau_2}\right) \right].$$

Since the contraction $\Delta\varepsilon$ represents, by definition, the negative difference between the elongation at the moment t and the initial elongation, from Eqs. (4) and (6) it follows

$$(7) \quad \Delta\varepsilon(t) = -[\varepsilon(t) - \varepsilon(0)] = -\frac{\sigma t}{\eta_3^*} + \left(\varepsilon_{20} - \frac{\sigma}{E_2} \right) \left[1 - \exp\left(-\frac{t}{\tau_2}\right) \right].$$

By analysing the Eq. (7), some aspects of the thermal stabilization process result. It is worth mentioning that (7) contains a positive component assigned to the contraction, and a negative one representing the contribution of the elongation due the tensile unitar stress σ .

In the absence of the stress applied to the fibre during the thermal stabilization ($\sigma=0$), the contraction $\Delta\varepsilon$ is always positive and monotonously increases with the time and the relaxation time $1/\varepsilon_2$, i.e.

$$(8) \quad \Delta\varepsilon(t) = \varepsilon_{20} \left[1 - \exp\left(-\frac{t}{\tau_2}\right) \right].$$

The application of a tensile force during the thermal treatment for stabilization leads to much more complex phenomena, namely, when the applied stress exceeds the product $\varepsilon_{20}E_2$, which equals the fibre internal stress before the stabilization, the elongation will decrease with the thermal treatment duration and the contraction $\Delta\varepsilon$ will be positive ($\Delta\varepsilon > 0$).

Yet, when the external stress is applied to a sample with a small internal stress, the sample elongates and $\Delta\varepsilon < 0$.

If the deformation takes place under a zero external stress ($\sigma=0$, free recovery), the final contraction will be equal to sample viscoelastic deformation.

Generally, the contraction $\Delta\varepsilon$ increases monotonously with the internal stress $\varepsilon_{20}E_2$ and decreases with the applied stress. This contraction is a

function of the stretching conditions and increases with increasing molecular orientation produced by the stretching process.

3. Experimental Technique. Polyethylene terephthalate (PET) fibres were used for study, with different birefringences ranging between $(4.2...22) \cdot 10^{-3}$. The samples were subjected to a thermal treatment with warm air. The temperature varied between 60 and 180°C . The thermal treatment was both stress free and under stress. The applied unitary stress changed within the range $5.09 \cdot 10^{10} \dots 18.84 \cdot 10^{13} \text{ N/m}^2$, depending on the number of the yarn monofilaments and on their diameters.

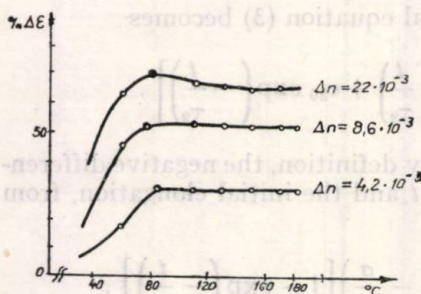


Fig. 2 — The free contraction vs. temperature for the preoriented PET fibres.

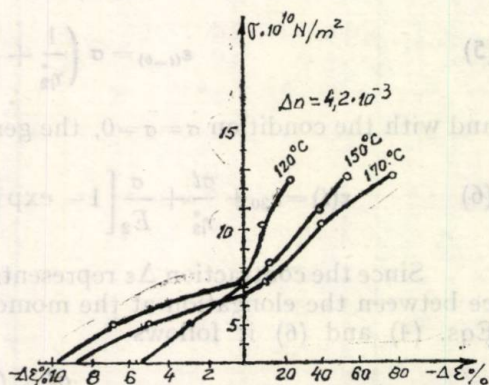


Fig. 3 — The dimensional variations of PET fibres subjected to a thermal treatment under stress, for $\Delta n = 4.2 \cdot 10^{-3}$.

4. Experimental Results and Discussion. 4.1. Presentation of the Results. In the graphics shown in Fig. 2, we present the free contraction for the three types of PET fibres we have used, whose pre-stress is given

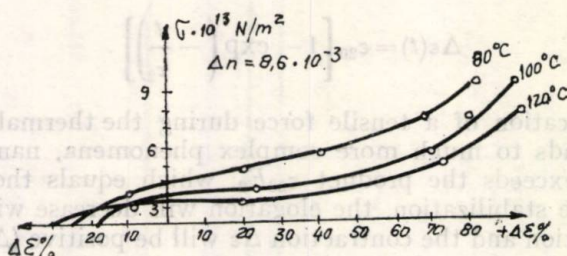


Fig. 4 — The dimensional variations of PET fibres subjected to a thermal treatment under stress, for $\Delta n = 8.6 \cdot 10^{-3}$.

by the birefringence values $4.2; 8.6$ and $22 \cdot 10^{-3}$. The curves presented in Figs. 3, 4 and 5 illustrate the dimensional change of the pre-stressed fibres subjected to a thermal treatment under stress.

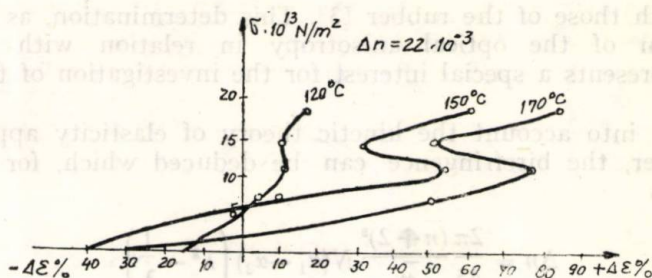


Fig. 5 — The dimensional variations of PET fibres subjected to a thermal treatment under stress for $\Delta n = 22 \cdot 10^{-3}$.

4.2. *Discussion of the Results.* From the examination of the curves obtained for the free contraction, one can see that the magnitude of the contraction increases with increasing temperature up to a maximum situated in the temperature range of 75...85°C and then decreases to a constant value. This maximum is more evident for preoriented samples, corresponding to the birefringence of $22 \cdot 10^{-3}$ and is connected with the starting point of the contraction mechanism. The result can be explained on the basis of the relation (8); in the conditions of a thermal treatment applied to a fibre in the absence of the external stress ($\sigma=0$), the contraction is always positive and equal to the viscoelastic component of the sample. When an external stress σ is applied the thermal stabilization of the fibres already having an orientation, made evident by the birefringence magnitude, one can see in Figs. 3, 4 and 5 that, in the case of a slight orientation, the retraction decreases with increasing temperature. The smaller is the birefringence, the more intense is the decrease.

Singular points registered for the fibres with the birefringence of $\Delta n = 22 \cdot 10^{-3}$ for temperatures exceeding 100°C (at 120°C an inflexion point is already seen and at 150°C and 170°C this inflexion is quite evident) indicate that for the partially oriented fibres the sample is subjected to a stretching process during the thermal stabilization under stress.

By intersecting these curves with the stress axes, the tensile stress can be determined, which is equal to the stress appeared in the fibre during the thermal treatment for a null elongation. The value of the tensile force increases with increasing molecular orientation (see Figs. 3–5) within the range $7.63 \cdot 10^{10} \dots 5 \cdot 10^{13} \text{ N/m}^2$.

It is worth mentioning that the slopes of the curves $\sigma=f(\Delta l)$ increase with decreasing temperature. This fact was made evident for PET fibres [2], [4].

Based on the theory of the viscoelastic materials [1], [5] applied for the viscoelastic linear model, in the hypothesis of the isothermal strain under stress, given by the relation (7) for the stretched fibres, the most important aspects of the thermal contractions observed on the preoriented polyestheric fibres can get a physical interpretation.

On the other side, it is well known that above the vitrifying point, the PET relatively amorphous fibres have optical and mechanical properties

identical with those of the rubber [3]. This determination, as well as the interpretation of the optical anisotropy in relation with the rubber elongation presents a special interest for the investigation of the polymer elasticity.

Taking into account the kinetic theory of elasticity applied to the elastic rubber, the birefringence can be deduced which, for the case of PET [8] is

$$(9) \quad \Delta n = \frac{2\pi (\bar{n}^2 - 2)^2}{45 \bar{n}} N(\alpha_1 - \alpha_2) \left(\lambda^2 - \frac{1}{\lambda} \right),$$

where $\bar{n} = 1.582$ for PET [6] and represents the average value of the sample refractive index equal to $(n_1 + 2n_2)/3$; $\Delta n = n_1 - n_2$, n_1 being the sample refractive index in the tensile direction and n_2 —in the direction normal to this; $N = 3.37 \cdot 10^{20}$ reticulate chains/cm³ and $\alpha_1 - \alpha_2 = 2.7 \cdot 10^{-23}$ cm³ at $T = 343^\circ\text{K}$ represents the difference of the molecular polarizabilities in the longitudinal (tensile) and transversal direction respectively [7]; λ represents the stretching ratio. With the obtained values [7], relation (9) becomes

$$(10) \quad \Delta n = \text{const.} \left(\lambda^2 - \frac{1}{\lambda} \right).$$

In the present work, the stretching ratio λ was deduced from the values of the maximum absolute contraction which are (see Fig. 2) as follows: for $\Delta n = 4.2 \cdot 10^{-3}$, the contraction $c = 0.32$; for $\Delta n = 8.6 \cdot 10^{-3}$, $c = 0.50$ and for $\Delta n = 22 \cdot 10^{-3}$, $c = 0.69$. By replacing these values in the relation $\lambda = 1/(1-c)$, one can get

$$(11) \quad \Delta n = \text{const.} \left[\frac{1}{(1-c)^2} - (1-c) \right],$$

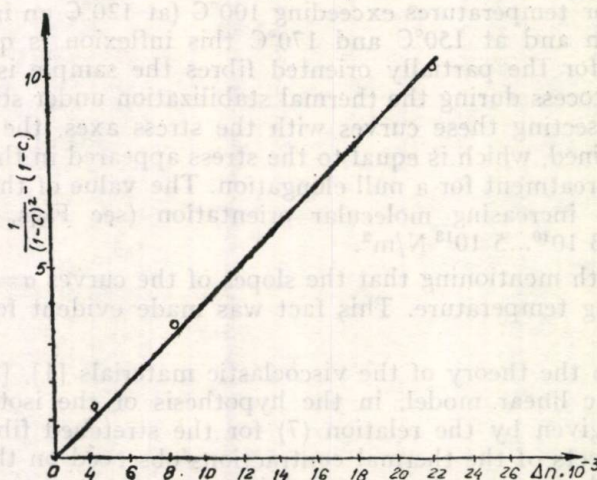


Fig. 6 — The contraction of PET filaments depending on the birefringence.

whose graphycal transposition from Fig. 6 leads to a line, this showing the proportionality between the birefringence and contraction.

Since the polymer is considered as consisting of a lattice of free-articulated chains identically connected, a correlation between the strain σ and the optical anisotropy can be established [3]. From the molecular theory of elasticity applied to the rubber, it is well known that the stress produced by stretching is

$$(12) \quad \sigma = NkT \left(\lambda^2 - \frac{1}{\lambda} \right),$$

where N and λ have the same meaning as in (9), and k is the Boltzman's constant. By deviding the relation (9) by (12), the stress-optical coefficient is obtained

$$(13) \quad \frac{\Delta n}{\sigma} = \frac{2\pi (\bar{n} + 2)^2}{45} \frac{\alpha_1 - \alpha_2}{\bar{n} kT} = \frac{\text{const.}}{T},$$

whence

$$(14) \quad \Delta n = \text{const.} \frac{\sigma}{T},$$

relation that reveals theoretically and experimentally seen decrease of the birefringence with increasing temperature.

During the thermal treatment, the molecules tend to become oriented perpendicularly to the fibre axis, resulting in the fibre contraction and birefringence decrease. In the case of thermal stabilization in air within the temperature range 75...85°C, the thermal contraction becomes maximum and the measured birefringence tends to zero.

5. Conclusions. The measurement of the fibre thermal contraction permits the determination of different parameters, like free contraction, tensile force a.s.o., parameters that characterize the stresses accumulated in the fibres during the technological processes.

If the sample is thermally stabilized at temperatures exceeding the vitrifying point, an increase of the molecular mobility takes place in the sample. In the absence of the external force, the energy dissipated in the yarn leads to dimensional instability or to contraction. If the yarn is maintained at a constant dimension during the thermal treatment, a „frozen“ stress appears in it, with its absolute value equal to the force that should be applied for keeping the stress constant, i.e. the retracting (contraction) force.

The maximum contraction stress is directly connected with the optical orientational anisotropy Δn .

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STUDIUL CONTRACȚIEI TERMICE

Metodă de estimare a acumulării tensiunilor interne în firele de polietilentereftalat (PET) în cursul proceselor tehnologice

(Rezumat)

Se determină contracția termică, elongația și birefringența firelor de polietilentereftalat (PET) supuse tratamentului termic cu aer cald în domeniul temperaturilor 60...180°C. Tratatamentul s-a aplicat atât firelor libere cât și celor tensionate la care s-a determinat coeficientul tensio-optic pentru probele studiate. Se conchide că prin simpla măsurare a contracției termice se pot estima tensiunile ce se acumulează în fibre în timpul proceselor tehnologice. Tensiunea acumulată este strins legată de anizotropia optică.

THE DESCRIPTION OF THERMOSTIMULATED PROCESSES BY MEANS OF THE SOLUTION OF A DEBYE TYPE EQUATION IN STRONG ELECTRIC FIELDS

THE INFLUENCE OF FORMING CONDITIONS

BY

RODICA NEAGU

0. Introduction. The method of thermally stimulated discharge, TSD, is a recent way proposed for a better understanding of the storage and decay of charge mechanisms in dielectrics. This method deals with the study of the decay of electret charge, as a result of its heating with a constant rate. The processes of the charge decrease are thus investigated as a function of temperature and consequently of time. The advantage of this technique is that it reveals the nature of the different processes of the stored charge decay in a short time and with a great sensibility and power of resolution.

One of the mechanisms that leads to the diminishing of the electrets charge is the dipolar relaxation. For the description of this process the classical Debye equation is used. This equation was obtained for the study of the behaviour of polar gases and dilute solutions of polar components in weak electric fields and then applied for the study of dipolar relaxation in the obtained electrets from polar polymers.

In a former paper we have proposed a correction of this equation, correction which allows the study of the thermostimulated processes due to dipolar relaxation for the obtained electrets out of polar polymers in strong electric fields $E > 4 \cdot 10^6$ V/m.

In this paper we find the solution of the Debye equation in strong electric fields. The solution gives the electret's polarization obtained at the end of the forming period. We compare this solution with that obtained from the equation used now in literature [2].

1. The Solution of the Debye Equation for Frozen Polarization Caused by Dipolar Orientation in Weak Electric Fields. Although the electrets obtained nowadays from polar polymers by field-temperature treatment, were realized in strong electric fields $E > 4 \cdot 10^6$ V/m, the most recent works, dealing with thermostimulated processes of these electrets, taking into account only the dipolar relaxation mechanisms, uses the classical Debye equation

$$(1) \quad \frac{dP}{dt} + \alpha(T) P(t) = \epsilon_0(\epsilon_s - \epsilon_\infty) \alpha(T) E$$

for the study of the electret's frozen polarization as a result of the forming process. Then

$$(2) \quad \frac{dP}{dt} + \alpha(T)P(t) = 0$$

is used, which is obtained from (1), in a TSD cycle conditions, i.e. when the shorted electret is heated with a constant rate $dT/dt=s$.

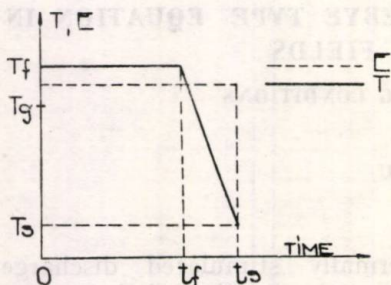


Fig. 1

The solution of Eq. (1) is obtained taking into account the conditions imposed by the experiment: the dielectric (polymer) is heated at a forming temperature T_f , during the t_f time, the temperature T_f being greater than the glass-rubber transition temperature. At the same time the forming field E , which is usually greater than 10 V/m, is applied during $t_f + t_s$. While the polymer is cooled up to the initial temperature of a TSD cycle, the forming field continues to be applied (Fig. 1). The

time of the cooling process is shorter than the forming time.

Eq. (1) is a differential linear one and its solution can be thus found in a usual way

$$P_s(t) = Ce^{-\int \alpha(T) dt} + \epsilon_0(\epsilon_s - \epsilon_\infty)E.$$

We obtain the value of the constant from the initial condition

$$t=0 \rightarrow P=0, \quad \text{i.e.}$$

$$P_s = P_s(t_s) = \epsilon_0(\epsilon_s - \epsilon_\infty) E \left[1 - e^{-\int_0^{t_s} \alpha(t) dt} \right].$$

It can be written also

$$P_s = P_s(t_s) = \epsilon_0(\epsilon_s - \epsilon_\infty) E \left[1 - e^{-\int_0^{t_f} \alpha(T_f) dt} e^{-\int_{t_f}^{t_s} \alpha(T) dt} \right],$$

but $T_f = \text{constant}$, thus $\alpha(T_f) = \text{constant}$ too, and

$$P_s = P_s(t_s) = \epsilon_0(\epsilon_s - \epsilon_\infty) E \left[1 - e^{-\alpha(T_f)t_f} e^{-\int_{t_f}^{t_s} \alpha(T) dt} \right].$$

This is the obtained polarization in electret at the end of the cooling period, or the electret's frozen polarization. Because the period in which the electret is cooled is very short in comparison with the forming time and because the polarization obtained after the orientation of the greatest number of dipoles in the heated dielectric placed in a strong electric field, can be modified very little during the cooling in field, the solution of Eq. (1) may be written, with a good approximation, as

$$(3) \quad P_s^{(1)} = P_s(t_f) = \epsilon_0(\epsilon_s - \epsilon_\infty) E [1 - e^{-\alpha(T_f)t_f}].$$

This is the solution used now by all the authors.

We suggest that this solution should be obtained in another way as well. We notice that the condition field-temperature has a special configuration looking like a step. As a consequence we consider that during the electret's forming cycle we can suppose with an unanimously accepted approximation, that the condition field-temperature is as in Fig. 2. In such configuration

$$T = T_f = \text{constant}, \quad E = \text{constant}.$$

Eq. (1) becomes

$$(4) \quad \frac{dP}{dt} + \alpha(T_f)P = \varepsilon_0(\varepsilon_s - \varepsilon_\infty)E,$$

where the coefficient of $P(t)$ and the free term are constant

$$B = -\alpha(T_f) = \text{constant}, \quad A = \varepsilon_0(\varepsilon_s - \varepsilon_\infty)E = \text{constant},$$

thus $dP/(A + BP) = dt$, equation with separable variables, which results in

$$(5) \quad P_s(t_f) = \varepsilon_0(\varepsilon_s - \varepsilon_\infty)E[1 - e^{-\alpha(T_f)t_f}].$$

We have found the same expression for the solution like that obtained by the usual method with a linear equation.

Because the equation proposed by us is of Riccati type it can be integrated only if we know a particular solution; we believe that this method of taking into account the particular condition of forming (step time-temperature-field) gives a possibility which leads to the solution.

2. The Solution of Debye Equation for the Frozen Polarization caused by the Dipolar Orientation in Strong Electric Fields. The Riccati's type equation obtained is based on the differential Debye equation for the distribution function f of the dipolar moments, in the general case of certain electric field, with varying intensity but constant direction:

$$\xi \frac{\partial f}{\partial t} = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left[\sin \theta \left(kT \frac{\partial f}{\partial \theta} \right) - Mf \right],$$

where $\xi/(2kT) = \tau$ is the dipolar relaxation time; θ — the angle between the applied field E and the dipolar moment of the molecule; $M = -\mu F \sin \theta$ — the torque; $F = E + P/(3\varepsilon_0)$ — the local Lorentz field.

If we consider that $X(t)$ is the orientational field at a certain moment, that acts on a dipole, we obtain the following set of Eqs. that describes the dipolar dielectric behaviour in strong fields:

$$\tau \frac{dX(t)}{dt} + X(t) = F(t) + \frac{\mu}{kT} X(t)F(t),$$

$$D = \varepsilon_0 E + P, \quad F = E + \frac{P}{3\varepsilon_0}, \quad P = \frac{n\mu^2}{3kT} X(t).$$

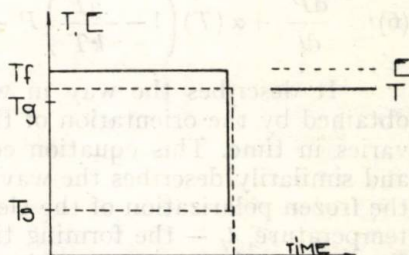


Fig. 2

By eliminating the functions F and X , we get the equation which describes the dielectric's behaviour composed of polar molecules under the influence of an external strong electric field:

$$(6) \quad \frac{dP}{dt} + \alpha(T) \left(1 - \frac{\mu E}{kT}\right) P - \frac{\mu}{3\epsilon_0 kT} \alpha(T) P^2 = \epsilon_0(\epsilon_s - \epsilon_\infty) \alpha(T) E.$$

It describes the way in which a part of the dielectric polarization obtained by the orientation of the dipolar moments in strong electric field, varies in time. This equation contains in the case of weak fields Eq. (1) and similarly describes the way in which the forming parameters influence the frozen polarization of the electret. If we consider that T_f is the forming temperature, t_f — the forming time, E — the forming field, the solution of Eq. (6) for these values of the variables gives the value of the dielectric polarization at the beginning of a TSD cycle.

Let deduce the solution of this equation starting from the supposition that the coefficients of P , P^2 , and the free term are constant in the particular case of obtaining the electrets:

$$\frac{dP}{dt} = \frac{\mu}{3\epsilon_0 kT} \alpha(T_f) P^2 - \alpha(T_f) \left(1 - \frac{\mu E}{kT}\right) P + \epsilon_0(\epsilon_s - \epsilon_\infty) \alpha(T_f) E,$$

or

$$(7) \quad \frac{dP}{dt} = CP^2 + BP + A,$$

where

$$(8) \quad C = \frac{\mu}{3\epsilon_0 kT} \alpha(T_f), \quad B = -\left(1 - \frac{\mu E}{kT}\right) \alpha(T_f), \quad A = \epsilon_0(\epsilon_s - \epsilon_\infty) E \alpha(T_f).$$

Riccati equation changes into an equation with separable variables,

$$(9) \quad \int_0^{P_s} \frac{dP}{CP^2 + BP + A} = \int_0^{t_f} dt.$$

The left side integral gives three primitives depending on the sign of

$$\Delta = 4AC - B^2 = \alpha^2(T_f) \left\{ -1 - \frac{\mu^2 E^2}{k^2 T^2} + \frac{2\mu E}{3kT} (3 + 2\epsilon_s - 2\epsilon_\infty) \right\}.$$

If we replace in this formula $\mu = 10^{-29}$ c.m., $\epsilon_s - \epsilon_\infty \simeq 2$ (for the majority of polymers), $k = 1.3 \cdot 10^{-23}$ J/K, $E = 7.5 \cdot 10^6$ V/m (values used by us in the experiment), $T = 400$ K and $\epsilon_0 = 8.8 \cdot 10^{-12}$ F/m, we get $\Delta < 0$ and thus

$$(10) \quad t_f = \frac{1}{\sqrt{-\Delta}} \ln \left| \frac{B + 2CP - \sqrt{-\Delta}}{B + 2CP + \sqrt{-\Delta}} \right| \Bigg|_0^{P_s},$$

or

$$(11) \quad P_s = \frac{(B + \sqrt{-\Delta}) \frac{B - \sqrt{-\Delta}}{B + \sqrt{-\Delta}} e^{t_f \sqrt{-\Delta}} - B + \sqrt{-\Delta}}{2C \left(1 - \frac{B - \sqrt{-\Delta}}{B + \sqrt{-\Delta}} e^{t_f \sqrt{-\Delta}} \right)}.$$

This is the accurate solution which we are proposing. In order to compare this solution (11) with one used in the present (5), we must obtain an approximation when neglecting $\mu^2 E^2 / (k^2 T^2)$ which is much smaller than the unity. We obtain another approximation if we develop in series the radicals in respect to the powers of $\mu E / (kT) < 1$ (of magnitude $1.4 \cdot 10^{-2}$).

In order to compare this solution with (5) we notice that

$$(12) \quad \frac{B - \sqrt{-\Delta}}{B + \sqrt{-\Delta}} e^{t_f \sqrt{-\Delta}} = \left(1 - \frac{\mu E}{kT} \right) \left(1 + \frac{\epsilon_s - \epsilon_\infty}{3} \right) \frac{\tau k T}{\mu E (\epsilon_s - \epsilon_\infty)} e^{t_f \sqrt{-\Delta}},$$

expression which is much greater than the unity

$$(13) \quad \left(1 - \frac{\mu E}{kT} \right) \left(1 + \frac{\epsilon_s - \epsilon_\infty}{3} \right) \frac{3kT}{\mu E (\epsilon_s - \epsilon_\infty)} e^{t_f \sqrt{-\Delta}} \gg 1.$$

We can write then

$$P_s = \frac{(B + \sqrt{-\Delta}) e^X - (B - \sqrt{-\Delta})}{-2C e^X},$$

where X stands for

$$X = \ln \frac{B - \sqrt{-\Delta}}{B + \sqrt{-\Delta}} + t_f \sqrt{-\Delta}.$$

It results that

$$P_s = -\frac{B + \sqrt{-\Delta}}{2C} + \frac{B - \sqrt{-\Delta}}{2C} e^{-X}.$$

Let us note with

$$I_1 = -\frac{B + \sqrt{-\Delta}}{2C}; \quad I_2 = \frac{B - \sqrt{-\Delta}}{2C}$$

and evaluate these quantities taking into account the last approximate solution and the notations (8):

$$I_1 = E \epsilon_0 (\epsilon_s - \epsilon_\infty),$$

$$I_2 = -\left[\frac{3\epsilon_0 k T}{\mu} - 3\epsilon_0 E - \epsilon_0 E (\epsilon_s - \epsilon_\infty) \right].$$

It finally results that the approximative solution may be written as

$$(14) \quad P_s^{(2)} = \epsilon_0 (\epsilon_s - \epsilon_\infty) - \left[\frac{3\epsilon_0 k T}{\mu} - 3\epsilon_0 E - \epsilon_0 E (\epsilon_s - \epsilon_\infty) \right] e^{-X},$$

or

$$\begin{aligned}
 P_s^{(2)} = & \epsilon_0(\epsilon_s - \epsilon_\infty) - \left\{ \left| \frac{3\epsilon_0 kT}{\mu} - 3\epsilon_0 E - \epsilon_0 E(\epsilon_s - \epsilon_\infty) \right| \times \right. \\
 (15) \quad & \times \left. \left[\frac{1 - \frac{\mu E}{kT} \left(1 + \frac{\epsilon_s - \epsilon_\infty}{3} \right)}{\frac{\mu E}{3kT}(\epsilon_s - \epsilon_\infty)} \right] \right\} e^{-\alpha(T_f)^{t_f} \left\{ 1 - \frac{\mu E}{kT} \left[1 + 2 \frac{\epsilon_s - \epsilon_\infty}{3} \right] \right\}}.
 \end{aligned}$$

The solution (5) of the equation, true in weak fields, is

$$P_s^{(1)} = \epsilon_0(\epsilon_s - \epsilon_\infty)E - \epsilon_0(\epsilon_s - \epsilon_\infty)E e^{-\alpha(T_f)^{t_f}}.$$

In these expressions the maximal possible polarization is $\epsilon_0(\epsilon_s - \epsilon_\infty)E$. Because $\mu E/(kT) \ll 1$, in order to compare $P_s^{(1)}$ with $P_s^{(2)}$ we must compare the order of magnitude of the expressions

$$M = \epsilon_0(\epsilon_s - \epsilon_\infty)E$$

and

$$N = \left[\frac{3\epsilon_0 kT}{\mu} - 3\epsilon_0 E - \epsilon_0 E(\epsilon_s - \epsilon_\infty) \right] \left[1 - \frac{1 - \frac{\mu E}{kT} \left(1 + \frac{\epsilon_s - \epsilon_\infty}{3} \right)}{\frac{\mu E}{3kT}(\epsilon_s - \epsilon_\infty)} \right].$$

Using the values (9) we obtain

$$M \simeq 2\epsilon_0 E, \quad N \simeq \epsilon_0 E.$$

It results that $N < M$.

This allows us to say that the frozen polarization calculated by means of the proposed equation is greater than the frozen polarization calculated with a Debye type equation. This increased value of the initial polarization in a TSD cycle is in accordance with the experimental data [5] and it influences the magnitude of the thermostimulated discharged current.

3. Conclusions. 1. The solution of Debye type equation in strong electric fields, deduced in a former paper [4], gives the polarization obtained in a polar dielectric, under the treatment field-temperature in electric fields stronger than $4 \cdot 10^6 \text{ V/m}$.

2. The value of the polarization obtained by us is greater than that obtained by solving the classical Debye equation and gives a better approximation of the experimentally obtained data found in literature.

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DESCRIEREA PROCESELOR TERMOSTIMULATE CU AJUTORUL SOLUȚIEI ECUAȚIEI DE TIP DEBYE ÎN CIMPURI ELECTRICE INTENSE

Influența condițiilor de formare

(Rezumat)

Soluția ecuației de tip Debye în cimpuri electrice intense, ecuație propusă într-o lucrare anterioară [4], descrie polarizația obținută într-un dielectric polar supus unui tratament cîmp-temperatură în cimpuri electrice mai mari decît $4 \cdot 10^6$ V/m. Valoarea polarizației obținute aici este mai mare decît cea rezultată prin rezolvarea ecuației clasice Debye. Această valoare aproximează mai bine, decît cea din literatură, datele obținute experimental.

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Soluția ecuației de tip Debye în cimpuri electrice intense, ecuația propusă într-o lucrare anterioară [1], permite determinarea condițiilor optime de încălzire pentru un tratament chimic termic în cimpuri electrice mari (de ordinul 10^7 V/m). Valoarea potențialului optim este determinată mai precis decât în rezultatele anterioare datorită condițiilor de încălzire și a valorii constante de timp, determinate din literatura de specialitate.

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CONSTRUCTION DES SYSTÈMES DE BASE DES SOLUTIONS POLYNOMIALES POUR UNE CLASSE D'ÉQUATIONS POLYLINÉAIRES D'ORDRE SUPÉRIEUR AUX OPÉRATEURS DIFFÉRENTIELS DIFFÉRENTS, DONT CERTAINS SONT LES OPÉRATEURS POLYVIBRANTS DE MANGERON

PAR

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Les auteurs dédient ce travail aux fondateurs de la théorie et des applications des systèmes polyharmoniques, polyvibrants et polycaloriques

1. Introduction Dans quelques-une de travaux concernant différentes extensions des équations polylinéaires, dûs surtout aux initiatives du premier des auteurs en collaboration avec d'autres chercheurs, [1]—[3], on a considéré différents problèmes „bien posés“ au sens d'Hadarnard pour les équations polylinéaires aux dérivées partielles de forme :

$$(1) \quad (M_{(nx)}^{(m)} + L_{(py)}^{(r)})^{(q)} u(x, y) = f(x, y),$$

où l'opérateur $M_{(nx)}^{(m)}$ est l'opérateur polyvibrant d'ordre $m^{(1)}$ qui porte sur l'en-

¹⁾ Les équations aux opérateurs polyvibrants $M_n^{(m)}$ ou bien $M_n^{m_1+m_2+\dots+m_n}$ à savoir :

$$M_n^{(m)} \equiv \partial^{mn} / \partial x_1^m \partial x_2^m \dots \partial x_n^m$$

et

$$M_n^{(m_1+m_2+\dots+m_n)} \equiv \partial^{n(m_1+m_2+\dots+m_n)} / \partial x_1^{m_1} \partial x_2^{m_2} \dots \partial x_n^{m_n},$$

dont le prototype est l'équation polyvibrante

$$(*) \quad M_n^{(m)} [M_n^{(m)} [u(x)] + A(x)u(x)] + p[B(x)M_n^{(m)} [u(x)] + C(x)u(x)] = 0$$

et le problème spectral qui s'y rattache : l'équation (*) et les conditions à la frontière

$$(**) \quad u(x)|_{\partial R} = 0, \quad M_n^{(1)} [u(x)]|_{\partial R} = 0, \dots, \quad M_n^{(m-1)} [u(x)]|_{\partial R} = 0$$

(ou bien d'autres conditions à la frontière ∂R qui en peuvent assurer l'unicité et l'existence de la solution), dont la nouveauté consiste entre autres dans le fait que $x = (x_1, x_2, \dots, x_n)$ et $R = \{a_j \leq x_j \leq b_j, j = 1, 2, \dots, n\}$, ont été appelées, à la suite des travaux de son auteur, [13]—[14], „équations de Mangeron“, [6]—[12]. Tout en soulignant le fait que l'on peut considérer des problèmes „bien posés“ pour les équations polyvibrantes en choisissant pour des frontières d'autres domaines que ceux bornés par les caractéristiques [15], tant que le fait que l'on a découvert nombre d'applications de ces équations [16]—[18], les auteurs sont convaincus que le champs des applications des équations polyvibrantes s'élargera sans cesse grâce à l'accroissement du nombre de modèles mathématiques de nouveaux phénomènes que l'on étudie de nos jours [19]—[20].

semble de variables $x=(x_1, x_2, \dots, x_n)$ et l'opérateur différentiel d'ordre r $L_{xy}^{(r)}$ porte sur un autre ensemble de variables $y=(y_1, y_2, \dots, y_p)$, m , r et q étant eux aussi des nombres entiers positifs, tandis que quelques autres travaux ont été dédiés à l'étude des équations polyvibrantes de forme

$$(2) \quad \mathcal{M}_s u(x) \equiv (M_n^{(s)} + \mathcal{M}_{s-1})u(x) = f(x),$$

ou bien de ses itérées

$$(3) \quad \mathcal{M}_s^{(N)} u(x) \equiv (M_n^{(s)} + \mathcal{M}_{s-1})^{(N)} u(x) = f(x),$$

où N est un nombre entier ≥ 1 et $f(x)$ est une fonction connue suffisamment lisse de variables réelles $x=(x_1, x_2, \dots, x_n)$, [21].

Tout récemment M. le Prof. D. Mangeron, a exposé à l'Université d'État de Campinas, nombre de propriétés de certains polynômes, appelés polynômes polynômiales, où bien polynômes P , reliés au développement $(x_1 + x_2 + \dots + x_n)^t$, que l'on peut appliquer pour la constructions des solutions de certaines classes d'équations polylinéaires.

Les auteurs, dont les deux premiers ont pris part à l'élaboration de la Note [22] construisent ci-dessous, tout en se limitant au cas de deux variables indépendantes, x et y , les systèmes de base des solutions polynomiales de toute une classe d'équations polylinéaires d'ordre supérieur, aux opérateurs différents, dont certains sont les opérateurs polyvibrants de Mangeron:

2. Les polynômes P au cas de deux variables indépendantes et quelques-unes de leurs propriétés. Prenons pour le prototype des équations polylinéaires mentionnées ci-dessus l'équation

$$(4) \quad \mathcal{K}u(x, y)(M_2^{(m)} + \Delta_m)u(x, y) = f(x, y),$$

où $M_2^{(m)}$ est l'opérateur polyvibrant d'ordre m à deux variables indépendantes, x et y , et l'opérateur Δ_m est défini comme suit :

$$\Delta_m = \frac{\partial^{2m}}{\partial x^{2m}} + \frac{\partial^{2m}}{\partial y^{2m}}.$$

On aboutit à la construction des systèmes de base des ensembles de solutions polynomiales des équations polylinéaires de type (4) en prenant pour le point de départ les polynômes P qui correspondent au cas de deux variables indépendantes et que l'on appelle afin de souligner ce cas particulier les polynômes B , à savoir

$$(5) \quad B_{r,s}^n(x, y) = \sum_{k=0}^{\left[\frac{n-s}{r} \right]} \binom{n}{rk+s} x^{n-rk-s} y^{rk+s},$$

qui se relie au développement

$$(x+y)^n = \sum_{s=0}^{r-1} B_{r,s}^n(x, y).$$

On peut vérifier que les polynômes B jouissent de tout un ensemble de propriétés remarquables, telles, par exemple, les propriétés de différentiabilité :

$$\begin{aligned}\frac{\partial}{\partial x} B_{r,s}^n(x, y) &= n B_{r,s}^{n-1}(x, y); \\ \frac{\partial}{\partial y} B_{r,s}^n(x, y) &= n B_{r,s-1}^{n-1}(x, y); \quad s \neq 0; \\ \frac{\partial}{\partial y} B_{r,0}^n(x, y) &= n B_{r,-1}^{n-1}(x, y),\end{aligned}$$

les relations de récurrence :

$$\begin{aligned}B_{r,s}^n(x, y) &= x B_{r,s}^{n-1}(x, y) + y B_{r,s-1}^{n-1}(x, y); \\ B_{r,0}^n(x, y) &= x B_{r,0}^{n-1}(x, y) + y B_{r,-1}^{n-1}(x, y),\end{aligned}$$

certaines relations d'intégration et d'autres encore.

3. Détermination des systèmes de base pour les solutions polynomiales de l'équation polynôme (4). Dans le but de déterminer à l'aide des polynômes B les systèmes de base des solutions polynomiales de l'équation (4), construisons les combinaisons linéaires des polynômes B qui suivent :

$$(6) \quad P_{m,s}^n(x, y) = B_{3m,s}^n(x, y) - B_{3m,m+s}^n(x, y)$$

et

$$(7) \quad Q_{m,v}^{n+2m}(x, y) = B_{3m,v}^{n+2m}(x, y) - B_{3m,3m+v}^{n+2m}(x, y).$$

Les combinaisons linéaires (6) et (7) permettent d'en déduire les théorèmes suivants :

Théorème 1. Les polynômes $P_{m,s}^n(x, y)$ constituent, pour $s = 0, 1, 2, \dots, 2m-1$, le système de base des solutions polynomiales de degré n de l'équation (4) si l'on y prend $f(x, y) = 0$. Ce système de base est constitué de $2m$ polynômes linéairement indépendants.

En effet, on aboutit à la démonstration de ce théorème si l'on applique les formules de différentiation ci-dessus tout en traitant séparément les cas $0 \leq s \leq m-1$ et $m \leq s \leq 2m-1$.

Théorème 2. Les polynômes $Q_{m,v}^{n+2m}(x, y)$ constituent, pour $v = 0, 1, 2, \dots$, les solutions particulières de l'équation (4) si l'on y prend $f(x, y) = \binom{n}{v} x^{n-v} y^v$.

Ce dernier théorème permet de conclure qu'à l'aide des combinaisons linéaires ou bien des séries de polynômes (7) on peut trouver les solutions particulières de l'équation (4) dans le cas où $f(x, y)$ est lui aussi un polynôme ou bien une fonction analytique.

4. Construction effective des systèmes de base. Considérons, pour fixer les idées, le cas où l'on a pris $m=2$ et par suite où l'on prend pour l'équation (4) la forme ci-dessous

$$(8) \quad \frac{\partial^4 u(x, y)}{\partial x^2 \partial y^2} + \frac{\partial^4 u(x, y)}{\partial x^4} + \frac{\partial^4 u(x, y)}{\partial y^4} = 0.$$

On en peut conclure donc par le suivant :

Théorème 3. *Le système de base des solutions polynomiales de degré n de l'équation polylinéaire (8) est constitué de quatre polynômes de degré n linéairement indépendants, exprimés par les polynômes B que voici :*

$$P_{2,0}^n(x, y) = B_{6,0}^n(x, y) - B_{6,2}^n(x, y),$$

$$P_{2,1}^n(x, y) = B_{6,1}^n(x, y) - B_{6,3}^n(x, y),$$

$$P_{2,2}^n(x, y) = B_{6,2}^n(x, y) - B_{6,4}^n(x, y),$$

$$P_{2,3}^n(x, y) = B_{6,3}^n(x, y) - B_{6,5}^n(x, y).$$

Remarques. 1. Les systèmes de base des solutions polynomiales donnés par (6) et (7), peuvent être utilisés pour la résolution de différents problèmes concernant l'équation (4).

2. Il serait intéressant pour certaines applications qui conduisent aux modèles mathématiques polylinéaires à trois variables indépendantes d'exposer dans une Note à venir les propriétés des polynômes trinomiales, ou bien des polynômes T et d'en déduire pour quelques-unes des équations polylinéaires de la classe considérée les systèmes de base des solutions polynomiales.

3. Les polynômes $B_{r,s}^n(x, y)$ satisfont à l'équation des ondes, et par suite à l'équation polyvibrante du premier ordre de Mangeron, étudiée dans ce cas particulier depuis longtemps. Naturellement, il s'agit seulement des cas où l'on a pris $s=0, 1, 2, \dots, r-1$.

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CONSTRUCȚIA SISTEMELOR DE BAZĂ ALE SOLUȚIILOR POLINOMIALE PENTRU
O CLASĂ DE ECUAȚII POLILINIARE DE ORDIN SUPERIOR CU OPERATORI
DIFERENȚIALI DIFERIȚI, DINTRE CARE UNII SÎNT OPERATORI
POLIVIBRANȚI AI LUI MANGERON

(Rezumat)

Folosind o nouă clasă de polinoame, autorii construiesc sistemele de bază ale soluțiilor polinomiale pentru un grup de ecuații poliliniare de ordin superior cu operatori diferențiali diferiți, dintre care unii sînt operatori polivibranți. Se realizează astfel o nouă cale de soluționare a problemei propuse, calea anterioară fiind cea de aplicare a metodei operaționale [3].

Cîteva observații de la finele lucrării sugerează unele posibilități de extindere a rezultatelor care se stabilesc.

A TRIANGLE INTERPOLANT WITH LINEAR COEFFICIENTS BASED ON THE FUNDAMENTAL BILINEAR INTERPOLANT OF MANGERON¹⁾

BY

GREGORY M. NIELSON

*This paper is devoted to the fiftieth
 anniversary of
 MANGERON EQUATIONS,
 called by him „Polyvibrating Equations“*

1. Introduction. The fundamental bilinear interpolant of Mangeron [2],

$$(1.1) \quad M[f](x, y) = (1-y)f(x, 0) + yf(x, 1) + (1-x)f(0, y) + xf(1, y) - \\ - (1-x)(1-y)f(0, 0) - x(1-y)f(1, 0) - y(1-x)f(0, 1) - xyf(1, 1),$$

which interpolates to f on the boundary of $R = \{(x, y) : 0 \leq x \leq 1; 0 \leq y \leq 1\}$, has had wide spread influence in the area of approximation of bivariate functions. While $M[f]$ and its generalization to include interpolation to normal derivatives serves well for a rectangular domain, many applications require approximations which are not inherently rectangular and consequently there is interest in developing analogous methods for other domains. Of particular interest, due to the application in finite element analysis and scattered data interpolation, is a triangular domain. In this report, we present methods for a triangular domain which are patterned after the following two equivalent properties that characterize $M[f]$.

Characterization 1. $M[f]$ is the unique function in $C^{2,2}(R)$ which lies in the kernel of the operator $\partial^4/\partial x^2 \partial y^2$ and interpolates to f on the boundary of R .

Characterization 2. Among all functions in $C^{2,2}(R)$ which interpolate to f on the boundary of R , $M[f]$ uniquely minimizes the pseudonorm

$$\sqrt{\iint_R [h_{xy}(s, t)]^2 ds dt}.$$

¹⁾ This research supported by the Office of Naval Research under Contract NR 044-433.

2. Linear Interpolation for Triangles. For notational convenience, we will use the standard triangular domain T with vertices $(0, 0)$, $(0, 1)$, and $(1, 0)$. Any other triangular domain can be mapped linearly to T by the use of an affine transformation. By $C^{n,m}(T)$, we mean those functions which possess continuous partial derivatives $\partial^{i+j}f/\partial x^i\partial y^j$; $i \leq n$, $j \leq m$ on T .

In the development of techniques of interpolation for triangular domains, there is a tendency towards the use of rational weight functions. This is due, in part, to the fact that M can be viewed as the boolean sum of two essentially univariate operators which are based upon linear interpolation along lines parallel to the sides of R . That is

$$M = M_x \oplus M_y = M_x + M_y - M_x M_y,$$

where

$$M_x[f](x, y) = (1-x)f(0, y) + xf(1, y),$$

$$M_y[f](x, y) = (1-y)f(x, 0) + yf(x, 1).$$

The corresponding linear interpolation operators for the domain T have rational weight function. For example,

$$T_x[f](0, y) = \frac{1-x-y}{1-y}f(0, y) + \frac{x}{1-y}(f(1-y, y))$$

is based upon linear interpolation along lines parallel to the x -axis and so it is clear that any scheme based upon the Boolean sum of these operators will have rational weight functions. We now proceed to develop a method which is void of rational weight functions.

Concerning the first type of characterization of $M[f]$, the general form of a function $k(x, y)$ which has the property that $\partial^4 k / \partial x^2 \partial y^2 = 0$ is

$$k(x, y) = g_1(x) + g_2(y) + yg_3(x) + xg_4(y)$$

for arbitrary, twice differentiable functions g_i , $i=1, 2, 3, 4$. If we now select g_i , $i=1, 2, 3, 4$ by forcing k to interpolate to f on the boundary of T , we obtain the equations

$$g_1(x) + g_2(0) + xg_4(0) = f(x, 0),$$

$$g_1(0) + g_2(y) + yg_3(0) = f(0, y),$$

$$g_1(x) + g_2(1-x) + (1-x)g_3(x) + xg_4(1-x) = f(x, 1-x).$$

Solving these equations, leads to

$$\begin{aligned} k[f](x, y) &= \frac{2-y-2x}{2(1-x)}f(x, 0) + \frac{2-2y-x}{2(1-y)}f(0, y) + \\ (2.1) \quad &+ \frac{(x+y)(3-x-y)-2}{2(1-x)(1-y)}f(0, 0) + \frac{x}{2(1-y)}[f(1-y, y) - f(1-y, 0)] + \\ &+ \frac{y}{2(1-x)}[f(x, 1-x) - f(0, 1-x)] + xy[g(x) - g(1-y)] \end{aligned}$$

which interpolates to f on the boundary of T for arbitrary g . In order to eliminate the rational weight functions, we choose

$$g(x) = \frac{1-2x}{2x(1-x)} [f(x, 1-x) - f(0, 1-x) - f(x, 0) + f(0, 0)]$$

which leads to the interpolation formula

$$(2.2) \quad B[f](x, y) = (1-y)f(x, 0) + (1-x)f(0, y) - (1-x-y)f(0, 0) + \\ + x[f(1-y, y) - f(1-y, 0)] + y[f(x, 1-x) - f(0, 1-x)].$$

It is interesting that this choice of g not only yields an interpolant with linear weights, but it also leads to an approximation which has a property analogous to the second characterization of $M[f]$.

Theorem 2.1. *Among all functions in $C^{2,2}(T)$ which interpolate to f on the boundary of T , $B[f]$ uniquely minimizes $\langle \cdot, \cdot \rangle$ where*

$$\langle h, g \rangle = \iint_T h_{xy}(s, t) g_{xy}(s, t) ds dt.$$

Proof. First let h be any function in $C^{2,2}(T)$ which is zero on the boundary of T ; then by using integration by parts we have

$$(2.3) \quad \langle h, B[f] \rangle = \int_0^1 \int_0^{1-t} h_{xy}(s, t) \frac{\partial^2 B[f]}{\partial s \partial t}(s, t) ds dt = \\ = \int_0^1 h_y(1-t, t) \frac{\partial^2 B[f]}{\partial s \partial t}(1-t, t) dt - \int_0^1 h_y(0, t) \frac{\partial^2 B[f]}{\partial s \partial t}(0, t) dt - \\ - \iint_T h_y(s, t) \frac{\partial^3 B[f]}{\partial s^2 \partial t}(s, t) ds dt,$$

where

$$(2.4) \quad \frac{\partial^2 B[f]}{\partial s \partial t}(s, t) = f_x(s, 1-s) - f_y(s, 1-s) - f_x(s, 0) + f_y(0, 1-s) + \\ + f_y(1-t, t) - f_x(1-t, t) - f_y(0, t) + f_x(1-t, 0).$$

Since $h(0, t) = 0$ we have that $h_y(0, t) = 0$ and from (2.4), we can see that $\frac{\partial^2 B[f]}{\partial s \partial t}(1-t, t) = 0$ and so the first two terms of the right side of (2.3) vanish. Using integration by parts again, we have



$$\begin{aligned}
\langle h, B[f] \rangle &= - \int_0^1 \int_0^{1-s} h_y(s, t) \frac{\partial^3 B[f]}{\partial s^2 \partial t} (s, t) dt ds = \\
&= - \int_0^1 h(s, 1-s) \frac{\partial^3 B[f]}{\partial s^2 \partial t} (s, 1-s) ds + \int_0^1 h(s, 0) \frac{\partial^3 B[f]}{\partial s^2 \partial t} (s, 0) ds + \\
&\quad + \iint_T h(s, t) \frac{\partial^4 B[f]}{\partial s^2 \partial t^2} (s, t) dt ds.
\end{aligned}$$

The first two terms are zero because $h=0$ on the boundary of T . The last term is zero because $\partial^4 B[f]/\partial s^2 \partial t^2$ is zero on T .

Let $g \in C^{2,2}(T)$ have the property that it interpolates to f on the boundary of T . Then,

$$\begin{aligned}
\langle g, g \rangle - \langle B[f] B[f] \rangle &= \langle g - B[f], g - B[f] \rangle + 2 \langle g - B[f], B[f] \rangle = \\
&= \langle g - B[f], g - B[f] \rangle \geq 0
\end{aligned}$$

and so we have established the minimum property of $B[f]$. In order to show uniqueness, we assume the existence of another minimizing interpolant, say $\bar{g}$, and consider the error $e = \bar{g} - B[f]$. Since both minimize the pseudo-norm, we have that

$$\langle e, e \rangle = \langle \bar{g} - B[f], \bar{g} - B[f] \rangle = 0,$$

which implies that $e_{xy} = 0$ on T . This, along with the fact that $e = 0$ on the boundary of T , implies that $e = 0$ on T which concludes the argument.

Corollary 2.2. *The operator B is exact for any function of the form*

$$g(x, y) = g_1(x) + g_2(y) + xg_3(y) + yg_3(1-x)$$

for arbitrary functions g_i , $i=1, 2, 3$.

Proof. Applying B directly and using

$$g(x, 0) = g_1(x) + g_2(0) + xg_3(0),$$

$$g(0, y) = g_2(y) + g_1(0) + yg_3(1),$$

$$g(0, 0) = g_1(0) + g_2(0),$$

$$g(1-y, y) - g(1-y, 0) = g_2(y) + g_3(y) - g_2(0) - (1-y)g_3(0),$$

$$g(x, 1-x) - g(0, 1-x) = g_1(x) + g_3(1-x) - g_1(0) - (1-x)g_3(1),$$

will yield this result.

In order to analyze the error and rate of convergence of this approximation, we introduce the triangular domain T_h with vertices $(0, 0)$, $(0, h)$ and $(h, 0)$ and the corresponding operator

$$B_h[f](x, y) = B[f(xh, yh)] \left(\frac{x}{h}, \frac{y}{h} \right) = \frac{1}{h} [(h-y)f(x, 0) + (h-x)f(0, y) -$$

$$-(h-x-y)f(0,0)+x[f(h-y,y)-f(h-y,0)]+y[f(x,h-x)-f(0,h-x)].$$

Theorem 2.3. For $f \in C^{1,1}(T_h)$

$$|B_h[f](x,y)-f(x,y)| \leq \frac{h^2}{2} \|f_{xy}\|_h, \quad (x,y) \in T_h,$$

where the norm is the uniform norm on T_h .

Proof. Since it is true in general for $f \in C^{1,1}(T_h)$ that

$$f(x,y)-f(x,0)-f(0,y)+f(0,0)=\int_0^x \int_0^y f_{xy}(s,t) ds dt,$$

it follows that

$$\begin{aligned} B_h[f](x,y) &= \frac{y}{h} \int_0^x \int_0^{h-x} f_{xy}(s,t) ds dt + \frac{x}{h} \int_0^{h-y} \int_0^y f_{xy}(s,t) ds dt - \\ &\quad - \int_0^x \int_0^y f_{xy}(s,t) ds dt. \end{aligned}$$

Therefore

$$\begin{aligned} &|B_h[f](x,y)-f(x,y)| \leq \\ &\leq \frac{xy(h-x)}{h} \|f_{xy}\|_h + \frac{x(h-y)y}{h} \|f_{xy}\|_h + xy \|f_{xy}\|_h = \frac{xy(3h-x-y)}{h} \|f_{xy}\|_h. \end{aligned}$$

This bound is maximized for $(x,y) \in T_h$ when $x=y=h/2$ to yield the desired result.

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INTERPOLANTUL TRIUNGIULAR CU COEFICIENȚI LINIARI BAZAT PE INTERPOLANTUL BILINIAR FUNDAMENTAL AL LUI MANGERON

(Rezumat)

Se studiază interpolarea triunghiulară cu coeficienți liniari bazată pe interpolantul biliniar fundamental al lui Mangeron, introdus în urmă cu o jumătate de secol.

Let $f(x)$ be a function defined on $[a, b]$. Then we can define $f(x)$ as follows:

$$f(x) = \begin{cases} f_1(x) & \text{if } x \in [a, c] \\ f_2(x) & \text{if } x \in (c, b] \end{cases}$$

where $f_1(x)$ and $f_2(x)$ are functions defined on $[a, c]$ and $(c, b]$ respectively. Since $f(x)$ is continuous at c , we have

$$\lim_{x \rightarrow c^-} f_1(x) = \lim_{x \rightarrow c^+} f_2(x) = f(c)$$

It follows that

$$f(x) = \begin{cases} f_1(x) & \text{if } x \in [a, c] \\ f_2(x) & \text{if } x \in (c, b] \end{cases}$$

$$f(x) = \begin{cases} f_1(x) & \text{if } x \in [a, c] \\ f_2(x) & \text{if } x \in (c, b] \end{cases}$$

Therefore

$$f(x) = \begin{cases} f_1(x) & \text{if } x \in [a, c] \\ f_2(x) & \text{if } x \in (c, b] \end{cases}$$

This result is well known for $f(x)$ which is continuous at c . The proof is as follows:

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VON G. G. LORENTZ

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A THEOREM IN THE THEORY OF GROUPS

BY

GH. COSTOVICI

1. We shall give a necessary and sufficient condition for the subgroup generated by the union of a subgroup with a subset to coincide with this union.

2. *Notations.* If G is a multiplicative group and A and B are subsets of G , then, $AB = \{ab; a \in A \text{ and } b \in B\}$, $B^{-1} = \{b^{-1}; b \in B\}$ and $[A \cup B] =$ the subgroup generated by $A \cup B$. With e denote the neutral element of G and $\Phi =$ the empty set.

3. *Theorem.* If G is a multiplicative group, A is a subgroup of G and B is a non-empty subset of G , then we have

(I) $\Leftrightarrow$ (II) and (III) $\Rightarrow$ (IV), where,

$$(I) \quad A \subseteq BB \subseteq A \cup B, AB = BA = B = B^{-1} \text{ and } e \in \overline{B},$$

$$(II) \quad [A \cup B] = A \cup B \text{ and } A \cap B = \Phi,$$

$$(III) \quad A = BB \text{ and } AB = B,$$

$$(IV) \quad BA = B \text{ and } B^{-1} = B.$$

Proof. (I) $\Rightarrow$ (II). $[A \cup B] = A \cup B$ results from $A \cup A^{-1} \cup B \cup B^{-1} = A \cup B$, $AA = A \subseteq A \cup B$, $BB \subseteq A \cup B$ and $AB = BA = B \subseteq A \cup B$. As regards $A \cap B = \Phi$, let $x \in A \cap B$. Then $x \in A$ and $x \in B = B^{-1}$ hence, $x \in A$ and $x^{-1} \in B$ so that $e = xx^{-1} \in AB = B$, contradiction.

(II) $\Rightarrow$ (I). We show that $AB = B$. It is clear that $B \subseteq AB$. If $z \in AB$, exist $a \in A$ and $b \in B$ so that $z = ab$. $ab \in A$ because $ab = a_1 \in A$ would imply $b = a^{-1}a_1 \in A \cap B$, contradiction. Therefore $z = ab \in B$. The proof for $BA = B$ follows same way. We show that $B^{-1} = B$. If $b^{-1} \in B^{-1}$, therefore $b \in B$, then $b^{-1} \in A$ because $b^{-1} = a \in A$ implies $b = a^{-1} \in A \cap B$, contradiction. Therefore $b^{-1} \in B$ so that $B^{-1} \subseteq B$. If $b \in B$, we have $b = (b^{-1})^{-1} = b_1^{-1}$ with $b_1 \in B$. Therefore $b \in B^{-1}$ and so $B \subseteq B^{-1}$. We show that $A \subseteq BB$. If $a \in A$, as $AB = B \neq \Phi$, we have $ab' = b''$ with some b' and b'' in B . Therefore $a = b''(b')^{-1} \in BB$. $e \in B$, otherwise $e \in A \cap B$, contradiction.

(III) $\Rightarrow$ (IV). $A=BB$ and $AB=B$ implies $AB=BBB=BA=B$. If $b^{-1} \in B^{-1}$, therefore $b \in B$, we have $bb=a \Rightarrow b^{-1}b^{-1}=a^{-1} \in A \Rightarrow b^{-1}=a^{-1}b \in AB=B$ hence $B^{-1} \subseteq B$. If $b \in B$, then $b=(b^{-1})^{-1}=(b_1)^{-1}$ with $b_1 \in B^{-1}$ and then $B \subseteq B^{-1}$.

4. *Examples.* a) G = the multiplicative group of the non-zero reals, A = the multiplicative subgroup of the non-zero rationals and B = the set of the irrationals. Here A is strictly included in BB .

b) G = the additive group of the reals, A = the additive subgroup of the rationals and B = the set of the irrationals. Similarly A is strictly included in $B+B$.

c) Example for which $A=BB$ is the group with two elements.

d) Another examples and counter-examples can be extracted from the below Cayley's table. We consider the following matrices;

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad a = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad a_1 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$

$$b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad c_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix},$$

where $i^2 = -1$, and construct, by multiplication, the following Cayley's table:

	e	e_1	a	a_1	b	b_1	c	c_1
e	e	e_1	a	a_1	b	b_1	c	c_1
e_1	e_1	e	a_1	a	b_1	b	c_1	c
a	a	a_1	e_1	e	c	c_1	b_1	b
a_1	a_1	a	e	e_1	c_1	c	b	b_1
b	b	b_1	c_1	c	e_1	e	a	a_1
b_1	b_1	b	c	c_1	e	e_1	a_1	a
c	c	c_1	b	b_1	a_1	a	e_1	e
c_1	c_1	c	b_1	b	a	a_1	e	e_1

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O TEOREMĂ ÎN TEORIA GRUPURILOR

(Reznmat)

Se demonstrează o condiție necesară și suficientă ca subgrupul generat de reuniunea unui subgrup cu o submulțime să coincidă cu această reuniune.

THE STABILITY ON THE SPACE C_g FOR VOLTERRA INTEGRAL EQUATION

BY

ADRIAN CORDUNEANU

In this paper, some variation of constants formula are applied to the study of the stability on the function Banach Space C_g , for nonlinear Volterra integral equation. It is well-known that under assumption that the solution $x=x(t)$ of the equation

$$(1) \quad x(t) = f(t) + \int_0^t a(t, s)[x(s) + H(s, x(s))]ds, \quad t \geq 0$$

exists for $t \geq 0$, the following representation formula is true

$$(2) \quad x(t) = y(t) + \int_0^t r(t, s)H(s, x(s))ds, \quad t \geq 0.$$

Here $f=f(t)$ and $H=H(s, x)$ are continuous n -vector functions defined for $t \geq 0$ and respectively $s \geq 0, |x| < \infty$, $a=a(t, s)$ is a continuous $n \times n$ matrix-function defined for $0 \leq s \leq t < \infty$, $y=y(t)$ is the solution and $r=r(t, s)$ is the resolvent kernel of the linear equation

$$(3) \quad y(t) = f(t) + \int_0^t a(t, s)y(s)ds, \quad t \geq 0.$$

Of course, $x=x(t)$ and $y=y(t)$ are n -vector functions defined for $t \geq 0$. The resolvent kernel $r=r(t, s)$ may be defined as the unique solution of the functional equation

$$(4) \quad r(t, s) = a(t, s) + \int_s^t a(t, \tau)r(\tau, s)d\tau, \quad 0 \leq s \leq t < \infty.$$

In what follows, we suppose that the solution $x=x(t)$ of (1) is unique and exists for $t \geq 0$ and we will study its behaviour when $t \rightarrow \infty$, with respect to the „initial function“ $f=f(t)$. We also suppose that $H(s, 0)=0$ for $s \geq 0$.

Definition 1. C_g is the Banach space consisting of all continuous n -vector functions $x=x(t)$ defined on $[0, \infty)$, such that

$$(5) \quad \|x\|_g = \sup_{t \geq 0} \frac{|x(t)|}{g(t)} < \infty,$$

where $g=g(t)$ is a given positive function, continuous for $t \geq 0$.

Here $|\cdot|$ stands for an euclidean norm in R^n . For the square $n \times n$ matrix $a=(a_{ij})$, we will use the norm $|a| = \sup |a\xi|$ for $|\xi| = 1$, $\xi \in R^n$.

Definition 2. We say that the solution $\bar{x}(t)=0$ of (1) ($\bar{f}(t)=0$) is stable on C_g , if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $\|f\|_g < \delta(\varepsilon)$, then $\|x\|_g < \varepsilon$.

Definition 3. We say that the solution $\bar{x}(t)=0$ of (1) ($\bar{f}(t)=0$) is asymptotically stable on C_g , if it is stable and if

$$(6) \quad \lim_{t \rightarrow \infty} \frac{|x(t)|}{g(t)} = 0, \text{ provided that } \lim_{t \rightarrow \infty} \frac{|f(t)|}{g(t)} = 0.$$

Proposition 1. Assume that

$$(7) \quad \sup_{t \geq 0} \int_0^t |r(t, s)| \frac{g(s)}{g(t)} ds = R < \infty;$$

$$(8) \quad |H(s, x)| \leq L |x| \text{ for } s \geq 0, |x| < \infty, \text{ where } 0 < L < R^{-1}.$$

Then the solution $\bar{x}(t)=0$ of (1) ($\bar{f}(t)=0$) is stable on C_g .

Proof. Because

$$(9) \quad y(t) = f(t) + \int_0^t r(t, s) f(s) ds, \quad t \geq 0$$

we have, using the condition (7), the following obvious inequality

$$(10) \quad \|y\|_g \leq \|f\|_g + R \|f\|_g = (1+R) \|f\|_g.$$

From (2), we obtain for the solution $x=x(t)$ of (1)

$$(11) \quad \begin{aligned} \frac{|x(t)|}{g(t)} &\leq \frac{|y(t)|}{g(t)} + \int_0^t |r(t, s)| \frac{g(s)}{g(t)} \frac{|x(s)|}{g(s)} L ds \leq \\ &\leq (1+R) \|f\|_g + LR \|x\|_g, \quad t \geq 0, \end{aligned}$$

which implies that $\|x\|_g \leq A \|f\|_g$, where $A = (1+R)(1-LR)^{-1}$ is a constant. This shows that the solution $\bar{x}(t)=0$ of (1), corresponding to $\bar{f}(t)=0$, is stable on C_g in the sense of the given above Definition 2.

Proposition 2. Assume that the condition (7) is satisfied and that

$$(12) \quad |r(t, s)| \leq M \frac{g(t)}{g(s)} \text{ for } 0 \leq s \leq t < \infty, \text{ where } M > 0 \text{ is a constant};$$

$$(13) \quad |H(s, x)| \leq h(s) |x| \text{ for } s \geq 0, \quad \|x\| < \infty, \text{ where } \alpha = \int_0^{\infty} h(s) ds < \infty.$$

Then the solution $\bar{x}(t) = 0$ of (1) ($\bar{f}(t) = 0$) is stable on C_g . If in addition

$$(14) \quad \lim_{t \rightarrow \infty} \int_0^T |r(t, s)| \frac{g(s)}{g(t)} ds = 0 \text{ for every } T > 0 \text{ fixed}$$

then this solution is asymptotically stable on C_g .

Proof. From the integral representation (2), we obtain for the solution of (1) the following majoration

$$(15) \quad \frac{|x(t)|}{g(t)} \leq \frac{|y(t)|}{g(t)} + \int_0^t |r(t, s)| \frac{g(s)}{g(t)} \frac{|x(s)|}{g(s)} h(s) ds, \quad t \geq 0,$$

which implies, accordingly to (10) and (12), that

$$(16) \quad \frac{|x(t)|}{g(t)} \leq (1+R)\|f\|_g + M \int_0^t h(s) \frac{|x(s)|}{g(s)} ds, \quad t \geq 0.$$

Now, from Gronwall–Bellman lemma, it follows

$$(17) \quad \|x\|_g \leq B \|f\|_g, \quad \text{where } B = (1+R) \exp(M\alpha)$$

and the first part of the conclusion holds. For the second part of the proof, we fix the function $f \in C_g$ and we write for $t \geq 0$ the inequality

$$(18) \quad \frac{|x(t)|}{g(t)} \leq \frac{|f(t)|}{g(t)} + \int_0^t |r(t, s)| \frac{g(s)}{g(t)} \frac{|f(s)|}{g(s)} ds + \int_0^t |r(t, s)| \frac{g(s)}{g(t)} \frac{|x(s)|}{g(s)} h(s) ds.$$

Let $\varepsilon > 0$ be given; we find $T = T(\varepsilon)$ such that

$$(19) \quad \int_T^{\infty} h(s) ds \leq \frac{\varepsilon}{MB\|f\|_g} \quad \text{and} \quad \frac{|f(t)|}{g(t)} \leq \min \left\{ \frac{\varepsilon}{R}, \varepsilon \right\} \quad \text{if } t \geq T.$$

On the other hand, we can write

$$(20) \quad \int_0^T |r(t, s)| \frac{g(s)}{g(t)} ds \leq \frac{\varepsilon}{\|f\|_g}, \quad \text{if } t \geq T_1 = T_1(\varepsilon)$$

if we take into account the condition (14). We also have

$$(21) \quad \int_T^t |r(t, s)| \frac{g(s)}{g(t)} \frac{|f(s)|}{g(s)} ds \leq \left(\sup_{s \geq T} \frac{|f(s)|}{g(s)} \right) \int_0^t |r(t, s)| \frac{g(s)}{g(t)} ds \leq \varepsilon$$

for every $t \geq T$. From the relation

$$(22) \quad \int_0^T |r(t, s)| \frac{g(s)}{g(t)} \frac{|x(s)|}{g(s)} h(s) ds \leq B \|f\|_g \left(\sup_{0 \leq s \leq T} h(s) \right) \int_0^T |r(t, s)| \frac{g(s)}{g(t)} ds$$

we deduce that his first term is smaller than ε , provided that $t \geq T_2 = T_2(\varepsilon)$. We also easily deduce that

$$(23) \quad \int_T^t |r(t, s)| \frac{g(s)}{g(t)} \frac{|x(s)|}{g(s)} h(s) ds \leq MB \|f\|_g \int_T^t h(s) ds \leq \varepsilon$$

for every $t \geq T$. Starting from (18), we can write finally

$$(24) \quad \frac{|x(t)|}{g(t)} \leq 5\varepsilon \quad \text{if } t \geq T^* = T^*(\varepsilon),$$

where $T^*(\varepsilon) = \max\{T(\varepsilon), T_1(\varepsilon), T_2(\varepsilon)\}$. This concludes the proof.

Example. If $|r(t, s)| \leq M \exp(-b(t-s))$ for $0 \leq s \leq t < \infty$ and $g(t) = \exp(-at)$ with $a < b$, then the conclusion of the preceding Proposition 2 holds.

Remark 1. The integral formula (2) does not prove implicitly the existence of the solution $x=x(t)$ for (1), but may be used under suitable conditions in this purpose. Thus, if the restrictions (8) and (13) imposed on $H=H(s, x)$ are strengthened and replaced by

$$(8') \quad |H(s, x) - H(s, y)| \leq L |x - y| \quad \text{for } s \geq 0, \quad |x|, |y| < \infty$$

and respectively

$$(13') \quad |H(s, x) - H(s, y)| \leq h(s) |x - y| \quad \text{for } s \geq 0, \quad |x|, |y| < \infty,$$

then it follows that (1) has a unique continuous solution $x=x(t)$ defined for $t \geq 0$.

Remark 2. The inequality (17) shows that the assumption that $H=H(s, x)$ is defined for $s \geq 0, |x| < \infty$ may be weakened; it suffices to suppose that H is defined for $s \geq 0, |x| < b$ where b is finite.

Remark 3. Similar propositions may be formulated in the case of the integro-differential equation

$$(25) \quad \dot{x}(t) = f(t) + a(t)x(t) + \int_0^t b(t, s)x(s)ds + H(t, x(t)), \quad x(0) = x_0,$$

where x , f and H are n -dimensional vectors, $a=a(t)$ and $b=b(t, s)$ are $n \times n$ matrix-functions defined for $t \geq 0$ and $0 \leq s \leq t < \infty$ respectively, because if $R=R(t, s)$ satisfies the conditions

$$(26) \quad \begin{cases} \frac{\partial R}{\partial t}(t, s) = a(t)R(t, s) + \int_s^t b(t, \tau)R(\tau, s)d\tau, \\ R(s, s) = I, \quad 0 \leq s \leq t < \infty, \end{cases}$$

where I is the identity matrix, then for the solution $x=x(t)$ of (25) the following variation of constants formula is true

$$(27) \quad x(t) = y(t) + \int_0^t R(t, s) H(s, x(s)) ds, \quad t \geq 0,$$

where $y=y(t)$ is the solution of the problem

$$(28) \quad \dot{y}(t) = f(t) + a(t)y(t) + \int_0^t b(t, s)y(s)ds, \quad y(0) = x_0$$

and is given by

$$(29) \quad y(t) = R(t, 0)x_0 + \int_0^t R(t, s)f(s)ds, \quad t \geq 0.$$

We now consider a more general integral equation

$$(30) \quad x(t) = f(t) + \int_0^t [a(t, s)x(s) + H(t, s, x(s))]ds, \quad t \geq 0,$$

where x , f , H are n -vector functions and $a=a(t, s)$ is a $n \times n$ matrix-function and we pose the same question about the stability of the solution $\bar{x}(t)=0$, corresponding to the initial function $\bar{f}(t)=0$. Of course, it is assumed that $H(t, s, 0) = 0$ for $0 \leq s \leq t < \infty$.

Denote by $U = U(t, s)$ the matrix which solves the equation

$$(31) \quad U(t, s) = I + \int_s^t a(t, \tau)U(\tau, s)d\tau, \quad 0 \leq s \leq t < \infty,$$

where again I is the identity matrix and suppose that $f=f(t)$ is continuously differentiable and that $H(t, s, x)$ together with $H_t(t, s, x)$ are continuous for $0 \leq s \leq t < \infty$ and $|x| < b$. Then the following formula is true for the solution of (30),

$$(32) \quad x(t) = y(t) + \int_0^t U(t, s) \left[\frac{d}{ds} H(s, \tau, x(\tau)) d\tau \right] ds, \quad t \geq 0,$$

where $y=y(t)$ is given by

$$(33) \quad y(t) = U(t, 0)f(0) + \int_0^t U(t, s)f'(s)ds, \quad t \geq 0.$$

Definition 4. C_g^1 is the Banach space consisting of all continuously differentiable n -vector functions $f=f(t)$ defined on $[0, \infty)$, such that $f'(t)$ belongs to C_g . The norm in this space is given by

$$(34) \quad \|f\|_g^* = \max \left\{ \frac{|f(0)|}{g(0)}, \|f'\|_g \right\}.$$

Definition 5. We say that the solution $\bar{x}(t)=0$ of (30) ($\bar{f}(t)=0$) is stable on C_g^1 , if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $\|f\|_g^* < \delta(\varepsilon)$, then $\|x\|_g < \varepsilon$.

Proposition 3. Assume that

$$(35) \quad \sup_{t \geq 0} \int_0^t |U(t, s)| \frac{g(s)}{g(t)} ds = N < \infty;$$

$$(36) \quad |U(t, s)| \leq M \frac{g(t)}{g(s)} \text{ for } 0 \leq s \leq t < \infty, \text{ where } M > 0 \text{ is a constant};$$

$$(37) \quad |H(t, t, x)| \leq h_1(t) |x| \text{ for } t \geq 0, |x| < b;$$

$$(38) \quad |H_t(t, s, x)| \leq h_2(t, s) |x| \text{ for } 0 \leq s \leq t < \infty, |x| < b;$$

$$(39) \quad \beta = \int_0^\infty h(t)dt < \infty, \text{ where } h(t) = h_1(t) + \int_0^t h_2(t, s) \frac{g(s)}{g(t)} ds.$$

Then the solution $\bar{x}(t)=0$ of (30) ($\bar{f}(t)=0$) is stable on C_g^1 . If in addition, for every $T > 0$ we have

$$(40) \quad \lim_{t \rightarrow \infty} \int_0^T |U(t, s)| \frac{g(s)}{g(t)} ds = 0$$

then it follows

$$(41) \quad \lim_{t \rightarrow \infty} \frac{|x(t)|}{g(t)} = 0, \text{ provided that } f(0)=0 \text{ and } \lim_{t \rightarrow \infty} \frac{|f'(t)|}{g(t)} = 0.$$

Proof. Firstly we remark that $y=y(t)$ given by (33) satisfies the obvious inequality $\|y\|_g \leq (M+N)\|f\|_g^*$. Using (32) we obtain for $t \geq 0$

$$(42) \quad \frac{|x(t)|}{g(t)} \leq (M+N)\|f\|_g^* + \int_0^t |U(t, s)| \left(|H(s, s, x(s))| + \int_0^s |H_s(s, \tau, x(\tau))| d\tau \right) ds.$$

If we denote by $u(t) = \sup (|x(s)|/g(s))$ for $0 \leq s \leq t$ and use the conditions (36)–(39), we have

$$(43) \quad u(t) \leq (M + N) \|f\|_g^* + M \int_0^t h(s) u(s) ds, \quad t \geq 0$$

which implies the inequality

$$(44) \quad \|x\|_g \leq C \|f\|_g^*, \quad \text{where } C = (M + N) \exp(M\beta).$$

For the second part of the conclusion, we fix the function $f \in C_g^1$ satisfying the conditions required in (41) and use the relations (32), (33) and (44). We will obtain the majoration

$$(45) \quad \frac{|x(t)|}{g(t)} \leq \int_0^t \left(|U(t, s)| \frac{g(s)}{g(t)} \frac{|f'(s)|}{g(s)} + C \|f\|_g^* h(s) \right) ds, \quad t \geq 0$$

and we will act as in the proof of the Proposition 2.

Remark 4. The above Proposition 3 is closely related to the Theorem 4.2. and Corollary 4.3 in [1]. The variation of constants formula (2) was applied to obtain various stability results by many authors (see [2] and the bibliography given there).

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STABILITATEA PE SPAȚIUL C_g PENTRU ECUAȚIA INTEGRALĂ VOLTERRA

(Rezumat)

Pentru ecuațiile neliniare (1) și (30) se formulează rezultate de stabilitate a soluției, obținute cu ajutorul formulelor de tip „variația constantelor”, numerotate (2) și (32) în textul articolului.

It is shown that the conditions (1) and (2) are necessary and sufficient for the function $f(z)$ to be analytic in the domain D .

$$(3) \quad f(z) = \sum_{n=0}^{\infty} a_n z^n \quad (z \in D)$$

which implies the function $f(z)$ is analytic in the domain D .

$$(4) \quad f(z) = \sum_{n=0}^{\infty} a_n z^n \quad (z \in D)$$

For the second part of the theorem, we let the function $f(z)$ be analytic in the domain D and let the conditions (1) and (2) be satisfied. Then the function $f(z)$ is analytic in the domain D .

$$(5) \quad f(z) = \sum_{n=0}^{\infty} a_n z^n \quad (z \in D)$$

and we will see in the proof of the theorem that the function $f(z)$ is analytic in the domain D .

Let us now consider the case when the function $f(z)$ is analytic in the domain D and the conditions (1) and (2) are satisfied. Then the function $f(z)$ is analytic in the domain D .

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NONDISSIPATIVE CONTINUOUS PERTURBATIONS OF COMPACT SEMIGROUPS

BY

I. I. VRABIE

1. Let X and Y be two real Banach spaces with the norms denoted by $|\cdot|$ and $|\cdot|$ respectively by $\|\cdot\|$ and assume that Y is continuously embedded in X . We are given a m -dissipative operator $A: D(A) \subset X \rightarrow 2^X$ whose restriction $A_Y: D(A_Y) \subset Y \rightarrow 2^Y$ is supposed to be also m -dissipative, and a continuous function $f: R_+ \times U \rightarrow Y$, where U is a nonempty and open subset in Y .

Denote by $S(t): \overline{D(A)} \rightarrow \overline{D(A)}$ the semigroup of nonlinear contractions generated by A on $\overline{D(A)}$ — the closure of $D(A)$ in X , and by $S_Y(t): \overline{D(A_Y)} \rightarrow \overline{D(A_Y)}$ the semigroup of nonlinear contractions generated by A_Y on $\overline{D(A_Y)}$ — the closure of $D(A_Y)$ in Y .

The aim of the present Note is to formulate a general existence theorem for a class of nonlinear evolution equations of the form:

$$(1) \quad \frac{du(t)}{dt} \in Au(t) + f(t, u(t)) \quad \text{for } 0 \leq t \leq T$$

$$u(0) = u_0 \in U \cap \overline{D(A_Y)},$$

in the case in which $S(t)$ is compact for all $t > 0$.

We begin with the basic notations and definitions which will be in effect in the sequel.

Definition 1. A function $f: R_+ \times U \rightarrow Y$ is called X -bounded continuous if for each subset $M \subset U \subset Y$ bounded in Y , the restriction f_M of f to $R_+ \times M$ is continuous from $R_+ \times M$ into Y in the topology induced by X on M and on Y , i.e., if for every $(t, x) \in R_+ \times M$ and every $\{(t_n, x_n)\}_n \in N \subset R_+ \times M$ with $\lim_{n \rightarrow \infty} t_n = t$ in R_+ , $\lim_{n \rightarrow \infty} x_n = x$ in X , it follows $\lim_{n \rightarrow \infty} f(t_n, x_n) = f(t, x)$ in X .

Definition 2. A function $u: [0, T] \rightarrow U \cap \overline{D(A)}$ is called integral solution for the problem (1) on $[0, T]$ if u is continuous from $[0, T]$ into X , $u(0) = u_0$ and

$$(2) \quad |u(t) - x|^2 \leq |u(s) - x|^2 + 2 \int_s^t \langle f(\theta) + y, u(\theta) - x \rangle d\theta,$$

for all $0 \leq s \leq t \leq T$ and all $(x, y) \in D(A) \times X$ with $y \in Ax$.

As usually, in Definition 2, $\langle \cdot, \cdot \rangle : X \times X \rightarrow R$ is defined by

$$\langle y, x \rangle = \sup \{x^*(y) : x^* \in G(x)\},$$

where $G : X \rightarrow 2^{X^*}$ is the duality mapping, X^* being the dual of X .

The general assumptions are the following :

(H₁) X and Y are two real Banach spaces with Y continuously embedded in X , and U is a nonempty, open set in Y . Moreover, X is reflexive and is the dual of a real Banach space Z with the embedding $X^* \subset Z$ continuous.

(H₂) (i) $A : D(A) \subset X \rightarrow 2^X$ is a m -dissipative operator generating a strongly continuous semigroup of nonlinear contractions $S(t) : \overline{D(A)} \rightarrow \overline{D(A)}$ with $S(t)$ compact for all $t > 0$.

(ii) There exists a restriction $A_Y : D(A_Y) \subset Y \rightarrow 2^Y$ of A to Y which is also m -dissipative.

(H₃) $f : R_+ \times Y$ is continuous and X -bounded continuous.

2. Our main result is

Theorem 1. Assume that (H₁), (H₂) and (H₃) are satisfied. Then, for each $u_0 \in U \cap \overline{D(A_Y)}$, there exists $T > 0$ such that the problem (1) has at least one integral solution $u : [0, T] \rightarrow U$. Moreover, if in addition $U = Y$ and for each $T > 0$ there exists $C_T > 0$ and $D_T \in R$ such that

$$(3) \quad \|f(t, v)\| \leq C_T \|v\| + D_T,$$

for all $0 \leq t \leq T$ and $v \in Y$, then the solution u may be continued on the whole positive half-axis.

The proof of Theorem 1, based on a general compactness argument developed previously by Baras [1] and Vrabie [5] will appear in Vrabie [12].

We also note that very closely related results have been obtained by Pazy [4], Pavel [2], Pavel and Vrabie [3] in the case in which A is linear, and by Vrabie [5] ... [11], in the case in which A is nonlinear.

3. Using Theorem 1, one may deduce a local existence result (which at our knowledge seems not yet quoted in the literature), for the following second-order parabolic equation with unilateral constraints on the boundary :

$$(4) \quad \begin{aligned} \frac{\partial u}{\partial t} &= \sum_{i,j} \frac{\partial}{\partial x_j} \left(a_{ij} \frac{\partial u}{\partial x_i} \right) + \sum_i b_i \frac{\partial u}{\partial x_i} + cu + f(t, x, u) \text{ a.e. on }]0, T[\times \Omega \\ &- \frac{\partial u}{\partial n} \in \beta(u) \text{ a.e. on }]0, T[\times \Gamma \end{aligned}$$

$$u(0, x) = u_0(x) \text{ a.e. on } \Omega,$$

where if $\Omega \subset R^n$ is a bounded domain whose boundary Γ is a C^∞ -manifold $a_{ij} \in C^1(\overline{\Omega})$, $b_i, c \in L^\infty(\Omega)$, $\beta \subset R \times R$ is a maximal monotone graph with $0 \in \beta(0)$, $u_0 \in L^\infty(\Omega)$, $f : R_+ \times \overline{\Omega} \times R \rightarrow R$ is continuous and

$$- \frac{\partial u}{\partial n} = \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_j} \cos(\vec{n}, \vec{e}_i),$$

$\vec{n}$ being the outward normal of Γ and $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ the canonical base in R^n .

The hypotheses we shall use are the following:

(I₁) $\Omega \subset R^n$ is a bounded domain whose boundary Γ is a C^∞ -manifold.

(I₂) (i) $\beta \subset R \times R$ is a maximal monotone graph with $0 \in \beta(0)$ and $j: R - [0, +\infty]$ is a lower-semicontinuous, convex function whose sub-differential ∂j coincides to β .

(ii) $a_{ij}C^1 \in (\overline{\Omega})$, $b_i, c \in L^\infty(\Omega)$ verify:

$$a_{ij} = a_{ji} \text{ for all } i, j \in \{1, 2, \dots, n\}$$

$$c(x) \geq c_0 > 0 \text{ a.e. on } \Omega.$$

Moreover, there exists $\alpha > 0$, such that

$$\sum_{i,j} a_{ij} y_i y_j \geq \alpha \sum_i |y_i|^2$$

for all $y = (y_1, y_2, \dots, y_n) \in R^n$ and all $x \in \overline{\Omega}$.

(I₃) $f: \overline{R_+} \times \Omega \times R \rightarrow R$ is continuous.

From Theorem 1 we get

Theorem 2. Assume that (I₁), (I₂) and (I₃) are satisfied. Then, for each $u_0 \in L^\infty(\Omega)$, there exists $T > 0$, such that the problem (4) has at least one integral solution $u: [0, T] \rightarrow L^\infty(\Omega)$ verifying:

$$(5) \quad u \in C(0, T; L^2(\Omega)), \quad u \in W^{1,2}(h, T; L^2(\Omega)) \text{ for all } 0 < h < T,$$

$$u(t) \in H^2(\Omega) \text{ a.e. on }]0, T[;$$

$$(6) \quad t^{1/2} \frac{\partial u}{\partial t} \in L^2(0, T; L^2(\Omega)),$$

$$(7) \quad \frac{1}{2} \sum_{i,j} \int_{\Omega} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx + \frac{1}{2} \int_{\Omega} c(x) |u(x)|^2 dx + \int_{\Gamma} j(u) ds \in L^1(0, T; R)$$

and if in addition $u_0 \in L^\infty(\Omega) \cap H^1(\Omega)$ and $j(u) \in L^1(\Gamma)$, then

$$(8) \quad \frac{\partial u}{\partial t} \in L^2(0, T; L^2(\Omega)),$$

$$(9) \quad \frac{1}{2} \sum_{i,j} \int_{\Omega} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx + \frac{1}{2} \int_{\Omega} c(x) |u(x)|^2 dx + \int_{\Gamma} j(u) ds \in L^\infty(0, T; R).$$

Moreover, if for each $T > 0$, there exists $K_T > 0$ and $H_T \in R$ such that

$$|f(t, x, v)| \leq K_T |v| + H_T,$$

for all $0 \leq t \leq T$, a.e. $x \in \Omega$ and all $v \in R$, then the solution u may be continued on the whole positive half-axis.

For the proof of Theorem 2, as well as for other examples, see Vrabie [12].

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PERTURBAȚII CONTINUE NEDISIPATIVE ALE
SEMIGRUPURILOR COMPACTE

(Rezumat)

Se formulează o teoremă generală de existență pentru o clasă de ecuații neliniare de evoluție de forma (1), în care semigrupul contracțiilor neliniare generate de A , este compact.

TOWARD THE UNIFICATION OF THE THEORY OF NONLINEAR SEMIGROUPS

BY

N. H. PAVEL

1. Introduction. The problem of the generation of nonlinear semi-groups by a dissipative (possible multivalued) operator, is one of the most interesting problems of nonlinear analysis. The main motivation of this assertion is that nonlinear semi-groups are a convenient approach for the treatment of many partial differential equations arising in physics.

The mathematical contribution of this work is the proof of Martin's theorem via Kobayashi's theorem (see sect. 2). This proof is important because is very simple and contributes to the unification of the theory of the generation of nonlinear semi-groups. Moreover, using this proof, the presentation of this theory becomes simpler and shorter with at least ten pages. This since the original proof of Martin (which takes about 15 pages) is very natural and concise, so that there is no any hope to be simplified and reduced in length.

2. Let X be a Banach space with norm $\|\cdot\|$ and $A, B \subset X \times X$. As usual, we define

$$Ax = \{y \in X; [x, y] \in A\}, \quad D(A) = \{x \in X; Ax \neq \emptyset\}$$

$$R(A) = \bigcup_{x \in D(A)} Ax, \quad A^{-1} = \{[y, x]; [x, y] \in A\}$$

$$\lambda A = \{[x, \lambda y]; [x, y] \in A\}, \quad \lambda \in \mathbb{R}$$

$$A + B = \{[x, y + z]; [x, y] \in A, [x, z] \in B\}.$$

One identifies the mappings with their graphs. A subset $B \subset X \times X$ is said to be dissipative if for each $\lambda > 0$ and $[x_i, y_i] \in B, i = 1, 2$ we have

$$\|x_1 - x_2\| \leq \|x_1 - \lambda y_1 - (x_2 - \lambda y_2)\|.$$

The subset B is said to be ω -dissipative if $A - \omega I$ is dissipative (where I is the identity and $\omega \in \mathbb{R}$).

A semi-group on a subset $D \subset X$ is a function S on $[0, \infty)$ which satisfies

$$(2.1) \quad S(t) \text{ maps } D \text{ into } D, \quad \forall t \geq 0;$$

$$(2.2) \quad S(t + \tau) = S(t)S(\tau), \quad t, \tau \geq 0;$$

$$(2.3) \quad \lim_{t \downarrow 0} S(t)x = S(0)x = x, \quad \forall x \in D.$$

If in addition

$$(2.4) \quad \|S(t)x - S(t)y\| \leq e^{\omega t} \|x - y\|, \quad \forall x, y \in D, t \geq 0$$

then S is said to be a semi-group of type ω on D .

If $z \in X$, denote by $d(z, D)$ the distance from the point z to the subset D .

The semigroup S is said to be differentiable at $t_0 \in (0, \infty)$ if the derivative of the function $t \rightarrow S(t)x$ at t_0 exists (for each $x \in D$). For an arbitrary $T > 0$, denote by $\{t_i^n\}$ the partition $\Delta_n = \{0 = t_0^n < t_1^n < \dots < t_{N_n}^n < T \leq t_{N_n}^n\}$ of the interval $[0, T]$ satisfying the condition

$$(2.5) \quad |\Delta_n| = \max_{1 \leq i \leq N_n} (t_i^n - t_{i-1}^n) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Let us assume that there are $\{x_i^n\} \subset D(A)$ and $\{p_i^n\}$ such that

$$\frac{x_i^n - x_{i-1}^n}{t_i^n - t_{i-1}^n} - p_i^n \in Ax_i^n, \quad i = 1, 2, \dots, N_n, \quad n \geq 1,$$

$$(2.6) \quad x_0^n \rightarrow x \quad \text{as } n \rightarrow \infty, \quad x \in X,$$

$$p_n = \sum_{i=1}^{N_n} \|p_i^n\| (t_i^n - t_{i-1}^n) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

The step function $u_n: [0, T] \rightarrow X$ defined by

$$(2.7) \quad u_n(t) = \begin{cases} x_0^n, & t = 0 \\ x_i^n, & t \in (t_{i-1}^n, t_i^n] \cap (0, T], \quad i = 1, \dots, N_n, \end{cases}$$

is said to be a DS-approximate solution of the Cauchy problem

$$(2.8) \quad (d/dt)u(t) \in Au(t), \quad t \in (0, T), \quad u(0) = x_0.$$

We now are prepared to state the following

Kobayashi's Theorem. If $A \subset X \times X$ is ω -dissipative and satisfies

$$(2.9) \quad \lim_{h \downarrow 0} \frac{1}{h} d[x, R(I - hA)] = 0, \quad x \in \overline{D(A)},$$

then A generates a semigroup $S(t)$ of type ω on D . Precisely, for each $T > 0$ there is a DS-approximate solution u_n defined by (2.7) such that

$$(2.10) \quad S(t)x = \lim_{n \rightarrow \infty} u_n(t), \quad \forall t \in [0, T], \quad x \in \overline{D(A)}.$$

Any other DS-approximate solution u_n satisfies also (2.10).

The semigroup S associated with A via the above theorem is said to be generated by A . The function $t \rightarrow S(t)x$ ($x \in \overline{D(A)}$) is the integral solution

of (2.8) (see e.g. [2]). In some particular cases, if the semigroup $S(t)$ is a.e. differentiable on $(0, +\infty)$, then $u(t) = S(t)x$ is just the strong solution of (2.8) ([1], [3]).

Recall also that in the particular case

$R(I - \lambda A) \supset \overline{D(A)}$, for all sufficiently small positive λ , $S(t)$ is given by the exponential formula of Crandall and Liggett [1]

$$S(t)x = \lim_{n \rightarrow \infty} \left(I - \frac{t}{n} A \right)^{-n} x, \quad x \in \overline{D(A)}, \quad t > 0.$$

In the continuous case, essentially the basic result is given by

Martin's Theorem [3]. *Let be a closed subset of X . If $A : D \rightarrow X$ is a continuous and dissipative operator satisfying (2.11), then the semigroup S generated by A is everywhere differentiable on $[0, \infty)$.*

Actually, in [3] Martin has obtained a result somewhat more general. There, the nonautonomous case is treated. Finally, in [3] the condition (2.9) appears under the equivalent form

$$(2.11) \quad \liminf_{h \downarrow 0} \frac{1}{h} d[x + hAx; D] = 0, \quad x \in D.$$

A proof of the equivalence of (2.9) and (2.11) even in more general conditions, may be found in [5].

We now proceed to show how Martin's result can be easily derived from that of Kobayashi.

Proof of Martin's theorem via Kobayashi's theorem. Let $x \in D$. Since A is continuous at x there are $M > 0$ and $R > 0$ such that

$$(2.12) \quad \|Au\| \leq M, \quad \forall u \in B(R),$$

where $B(R)$ is the ball of radius R about zero.

Let $T > 0$ with the property

$$(2.13) \quad T(M+1) < R.$$

Fix an arbitrary natural number n and set $t_0^n = 0$, $x_0^n = x$. Following Martin [3], inductively define t_{i+1}^n and x_{i+1}^n as follows: If $t_i^n = T$ set $t_{i+1}^n = T$ and if $0 < t_i^n < T$, define δ_i^n as the largest number $\left(0, \frac{1}{n}\right]$ satisfying

$$(2.14) \quad t_i^n + \delta_i^n \leq T;$$

$$(2.15) \quad \|Au - Ax_i^n\| \leq \frac{1}{n}, \quad \forall u \in D, \quad \|u - x_i^n\| \leq (M+1)\delta_i^n;$$

$$(2.16) \quad \frac{1}{\delta_i^n} d[x_i^n + \delta_i^n Ax_i^n; D] \leq \frac{1}{2n}.$$

In view of (2.11), $\delta_i^n > 0$. Since there is no danger of confusion we drop the index n , writing $t_i^n = t_i$, $x_i^n = x_i$ and $\delta_i^n = \delta_i$. Define

$$(2.17) \quad t_{i+1} = t_i + \delta_i.$$

According to (2.16), there is an element $x_{i+1} \in D$ such that

$$(2.18) \quad \frac{1}{\delta_i} \|x_i + \delta_i A x_i - x_{i+1}\| \leq \frac{1}{n}.$$

As in Pavel-Ursescu [5] (see also [4]), the following notation is useful

$$(2.19) \quad p_i = \frac{1}{\delta_i} (x_{i+1} - x_i - \delta_i A x_i), \quad i=0, 1, \dots$$

Thus we have

$$(2.20) \quad x_{i+1} = x_i + (t_{i+1} - t_i)(A x_i + p_i), \quad \|p_i\| \leq \frac{1}{n}.$$

It is easy to verify that $x_i \in B(K)$, hence (2.20) and (2.15) give

$$(2.21) \quad \|A x_{i+1} - A x_i\| \leq \frac{1}{n}.$$

It is also known that $\lim_{i \rightarrow \infty} t_i = T$ (see [3]).

Let $T_1 = T_2$ and $t \in (0, T_1]$. There is an integer $i = i(n, t)$ such that $t \in (t_i, t_{i+1}]$ and another one $i_n = N_n$ such that $t_{i_n} < T_1 \leq t_{i_n+1}$. Define $y_n : [0, T_1] \rightarrow X$ by

$$(2.22) \quad y_n(t) = x_i + (t - t_i)(A x_i + p_i), \quad t_i \leq t \leq t_{i+1}, \quad i=0, 1, \dots, N_n.$$

It is immediate that

$$\|y_n(t) - y_n(s)\| \leq (M+1)|t-s|, \quad t, s \in [0, T_1].$$

The crucial point is now to prove that the sequence (of polygonal lines) y_n is convergent. This is the difficult (and long) part of Martin's proof. To avoid it, let us observe that (2.20) yields

$$(2.23) \quad \frac{x_{i+1} - x_i}{t_{i+1} - t_i} - \varepsilon_i = A x_{i+1}, \quad i=0, 1, \dots, N_n$$

with

$$(2.24) \quad \varepsilon_i = \varepsilon_i^n = p_i + A x_i - A x_{i+1}.$$

By (2.17) and (2.18) we have $\|\varepsilon_i\| \leq 2/n$, therefore

$$(2.25) \quad \varepsilon^n = \sum_{i=1}^{\infty} \|\varepsilon_i^n\| (t_{i+1} - t_i) \leq \frac{T}{n}.$$

Consequently, the function $u_n : [0, T_1] \rightarrow D$ given by

$$(2.26) \quad u_n(t) = \begin{cases} x, & t=0 \\ x_{i+1}^n, & t_i < t \leq t_{i+1}, \quad i=0, 1, \dots, N_n \end{cases}$$

is a DS-approximate solution.

By Kobayashi's theorem, $\lim_{n \rightarrow \infty} u_n(t) = u(t) = S(t)x \in D$. On the other hand, it is easy to check that

$$(2.27) \quad \|y_n(t) - u_n(t)\| \leq \frac{2(M+1)}{n}, \quad \forall t \in [0, T_1].$$

Indeed, if $t=0$ then the left hand side of (2.27) is zero. If $t_i < t \leq t_{i+1}$, then we have

$$\|y_n(t) - u_n(t)\| = \|x_i - x_{i+1} + (t - t_i)(Ax_i + p_i)\| \leq [(t_{i+1} - t_i) + (t - t_i)](M+1).$$

This implies (2.27) since $t - t_i \leq t_{i+1} - t_i \leq 1/n$.

From (2.27) it follows that

$$(2.28) \quad \lim_{n \rightarrow \infty} y_n(t) = \lim_{n \rightarrow \infty} u_n(t) = S(t)x$$

uniformly on $[0, T]$.

Finally, the fact that $t \rightarrow u(t) = S(t)x$ is differentiable on $[0, T_1]$ it follows from the integral equation

$$(2.29) \quad u(t) = x + \int_0^t Au(s)ds, \quad t \in [0, T_1]$$

and from the continuity of A and of $t \rightarrow u(t)$. The equation (2.29) is obtained in Martin [3]. However, for the sake of simplicity we shall indicate here a proof of it, as in [5] or [4].

Define the step function

$$a_n(s) = \begin{cases} t_i, & t_i \leq s < t_{i+1} \\ T_1, & s = T_1. \end{cases}$$

It is easy to check that y_n can be written under the form

$$(2.30) \quad y_n(t) = x + \int_0^t Ay_n(a_n(s))ds + g_n(t),$$

where

$$g_n(t) = \sum_{j=0}^{i-1} (t_{j+1} - t_j)p_j + (t - t_i)p_i; \quad t_i \leq t \leq t_{i+1}$$

hence $\|g_n(t)\| \leq T_1/n$, $t \in [0, T_1]$. Since $y_n(a_n(s)) \rightarrow u(s)$ as $n \rightarrow \infty$, uniformly on $[0, T_1]$, (2.30) yields (2.29).

By standard arguments, one proves that the local solution u of (2.29) can be extended to the whole $[0, +\infty)$ and therefore (2.29) holds for all $t \geq 0$. This implies the differentiability of $t \rightarrow S(t)x$ everywhere on $[0, +\infty)$.

Remark 1. Similarly to (the „tangent condition on the right“) (2.11) let us consider the following „tangent condition on the left“

$$(2.31) \quad \liminf_{h \uparrow 0} \frac{1}{h} d(x + hAx; D) = 0, \quad x \in D$$

where $h \uparrow 0$ means $h \rightarrow 0$ with $h < 0$. For the existence of the solution $u: (-\infty, 0] \rightarrow D$ to the Cauchy problem

$$(2.32) \quad u'(t) = Au(t), \quad u(0) = x, \quad x \in D, \quad t \leq 0$$

the condition (2.31) is necessary. If $A : D \rightarrow X$ is Lipschitz continuous on D , then (2.31) is also sufficient (hence necessary and sufficient) for the existence of the solution $u(t)$ of (2.32).

Necessity. Since $u(h) = u(h, x) \in D$, $h < 0$, we have

$$\frac{1}{-h} d(x + hAx; D) \leq \frac{1}{-h} \|x + hAx - u(h)\| = \left\| \frac{u(h) - x}{h} - Ax \right\| \rightarrow 0 \text{ as } h \uparrow 0$$

which implies (2.31) with „lim“ instead of „lim inf“.

Sufficiency. Set $A_1 x = -Ax$, $x \in D$ and consider the problem

$$(2.33) \quad y'(s) = A_1(y(s)), \quad s \geq 0, \quad y(0) = x, \quad x \in D.$$

In view of (2.31), it follows that A_1 satisfies (2.11) and consequently the solution of (2.33) is just $y(s) = S(s)x$, $s \geq 0$. Now, the function $u = y(-t) = S(-t)x$, $t \leq 0$ is the solution to (2.32). It follows that if (2.11) holds and (2.31) is not true for $x_0 \in D$, then $S(t)x_0$, $t \geq 0$ cannot be extended to the left of zero (such that $S(t)x_0 \in D$ for $t < 0$). Moreover, if $A : D \rightarrow X$ is Lipschitz continuous on D , and both (2.11) and (2.31) are satisfied, then the semigroup S generated by A is defined on $R = (-\infty, +\infty)$ (i.e. is a group).

A simple example in this direction is the following one. Take $X = R$, $D = [0, 1] \subset R$ and $Ax = -x + 1$. It is easy to see that (2.11) holds while (2.31) is not satisfied for $x = 0$. Consequently, by the above theory $S(t)0$ does not belong to $[0, 1]$ for $t < 0$. Indeed, in this case $S(t)0 = 1 - e^{-t} \in [0, 1]$ for all $t \geq 0$ (but not for $t < 0$).

Remark 1 has been added after some conversations with D. Motreanu and G. Moroșanu.

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CONTRIBUȚII LA UNIFICAREA TEORIEI SEMIGRUPURILOR NELINIARE

(Rezumat)

Folosind soluții aproximative de tip „scheme cu diferențe“ se demonstrează că generarea și diferențiabilitatea semigrupurilor neliniare din cazul continuu, se poate deduce relativ simplu din cazul general (discontinuu). Acest fapt unifică (și simplifică cu câteva zeci de pagini) teoria respectivă. Se discută apoi condițiile în care semigrupurile generate se pot prelungi la grupuri.

POLYNOMIAL ALGEBRAS AND SYMBOLIC REPRESENTATION OF DISCRETE SIGNALS

BY

DAN IEL CONDURACHE

1. The notion of symbolic representation of a class of discrete signals on finite order algebras over the real number field was introduced in [2].

A $s(n)$ signal $s: \mathbb{Z} \rightarrow \mathbb{R}$ is representable symbolically on the finite order associative and commutative algebra over the real number field $\mathcal{A}$, by element $\hat{W} \in \mathcal{A}$, if the signal $s(n+1)$ is representable symbolically by element $\hat{D}\hat{W}$. $\hat{D}$ is a fixed element in the algebra with the role of a right-shift operator. The class of discrete signals representable symbolically by means of this proceeding is composed of the set of solutions to the homogeneous difference equation whose characteristic polynomial is the characteristic polynomial of element $\hat{D}$ in algebra $\mathcal{A}$ [2]. In what follows, we shall demonstrate that polynomial algebras enjoy some special properties from this point of view.

2. Let $\mathcal{A}_k$ be the polynomial algebra generated by a polynomial with real coefficients:

$$(1) \quad p(\lambda) = \lambda^k + a_1 \lambda^{k-1} + \dots + a_{k-1} \lambda + a_k; \quad a_i \in \mathbb{R}, \quad i = \overline{1, k}$$

and $\hat{v} \in \mathcal{A}$ the element generating a cyclical basis:

$$(2) \quad \{\hat{v}^0, \hat{v}, \dots, \hat{v}^{k-1}\}, \quad p(\hat{v}) = 0.$$

As $\hat{v}^n \in \mathcal{A}$, $n \in \mathbb{N}$, its unique denotation becomes:

$$(3) \quad \hat{v}^n = \varphi_1(n) \hat{v}^0 + \varphi_2(n) \hat{v} + \dots + \varphi_k(n) \hat{v}^{k-1}.$$

We shall prove the

Proposition 1. Functions $\varphi_i(n)$, $i = \overline{1, k}$ represent a fundamental system of solutions to homogeneous difference equation

$$(4) \quad \varphi_{n+k} + a_1 \varphi_{n+k-1} + \dots + a_k \varphi_n = 0.$$

Proof. Let $\mathcal{M}_k$ be the algebra of matrices isomorphic with the polynomial one $\mathcal{A}_k$ [4]. The matrix

$$\mathbf{V} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_k & -a_{k-1} & -a_{k-2} & \dots & a_1 \end{bmatrix}.$$

corresponds by isomorphism to element $\hat{v}$. That is why matrix $\mathbf{V}^n$ corresponds by isomorphism to element $\hat{v}^n$. This shall be denoted as

$$(5) \quad \hat{v}^n \Leftrightarrow \mathbf{V}^n.$$

Equation (4) is equivalent to the homogeneous first order difference system:

$$(6) \quad \begin{bmatrix} \varphi_1(n+1) \\ \vdots \\ \varphi_k(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_k & -a_{k-1} & \dots & -a_1 \end{bmatrix} \begin{bmatrix} \varphi_1(n) \\ \vdots \\ \varphi_k(n) \end{bmatrix}$$

or, using another denotation

$$(7) \quad \varphi(n+1) = \mathbf{V} \varphi(n).$$

The general solution to system (7) is

$$(8) \quad \varphi(n) = \mathbf{V}^n C,$$

where C is a column constant matrix. From (3), (5) and (8) it results the above enunciation.

Observation. Let $\{\hat{\varepsilon}_1, \hat{\varepsilon}_2, \dots, \hat{\varepsilon}_k\}$ be another basis in the polynomial algebra (2). In this basis we obtain

$$(9) \quad \hat{v}^n = \psi_1(n) \hat{\varepsilon}_1 + \psi_2(n) \hat{\varepsilon}_2 + \dots + \psi_k(n) \hat{\varepsilon}_k.$$

Functions $\psi_i(n)$, $i = \overline{1, k}$ also form a fundamental system of solutions to equation (4).

Indeed, let $\mathbf{A}$ be the change of basis matrix, i.e.:

$$(10) \quad [\hat{v}^0 \hat{v} \dots \hat{v}^{k-1}]^t = \mathbf{A} [\hat{\varepsilon}_1 \hat{\varepsilon}_2 \dots \hat{\varepsilon}_k]^t.$$

From (3), (9) and (10) it results

$$(11) \quad [\varphi_1(n) \varphi_2(n) \dots \varphi_k(n)]^t = \mathbf{A} [\psi_1(n) \psi_2(n) \dots \psi_k(n)]^t.$$

Matrix $\mathbf{A}$ being a non-singular one, we obtain from (11)

$$[\psi_1(n) \psi_2(n) \dots \psi_k(n)]^t = \mathbf{A}^{-1} [\varphi_1(n) \varphi_2(n) \dots \varphi_k(n)]^t.$$

fact showing that functions $\psi_i(n)$, $i = \overline{1, k}$ are linear combinations of functions $\varphi_i(n)$, $i = \overline{1, k}$ which form a fundamental system of solutions to equation (4), as it results from Proposition 1. As it follows, $\psi_i(n)$, $i = \overline{1, k}$ form a fundamental system of solutions. The observation becomes particularly useful when in basis $\{\hat{\varepsilon}_1, \hat{\varepsilon}_2, \dots, \hat{\varepsilon}_k\}$ an exponential form may be given to element $\hat{v}$:

$$\hat{v} = \exp(x_1 \hat{\varepsilon}_1 + x_2 \hat{\varepsilon}_2 + \dots + x_k \hat{\varepsilon}_k); \quad x_i \in \mathbb{R}, \quad i = \overline{1, k}.$$

In this case, a De Moivre's theorem like formula works in order to determine functions:

$$\hat{v}^n = \exp n(x_1 \hat{\varepsilon}_1 + x_2 \hat{\varepsilon}_2 + \dots + x_k \hat{\varepsilon}_k) = \psi_1(n) \hat{\varepsilon}_1 + \psi_2(n) \hat{\varepsilon}_2 + \dots + \psi_k(n) \hat{\varepsilon}_k.$$

3. We intend to find out which of the discrete signals are representable symbolically on the polynomial algebra $\mathcal{A}_k$ with the right-shift operator $\hat{v}$.

The signals representable symbolically by the elements of basis $\{\hat{v}^0, \hat{v}, \dots, \hat{v}^{k-1}\}$ are solutions to the first order homogeneous system denoted as

$$(12) \quad \varphi(n+1) = \mathbf{V} \varphi(n),$$

where $\mathbf{V}$ is a matrix belonging to the algebra of matrices isomorphic with $\mathcal{A}_k$, matrix corresponding by isomorphism to that element in the algebra which plays the role of a right-shift operator [2].

Under these circumstances, system (12) is identical to (7). Choosing in (8), $C]^t = [1, 0, \dots, 0]$ it results that the elements in the cyclical basis $\{\hat{v}^0, \hat{v}, \dots, \hat{v}^{k-1}\}$ in algebra $\mathcal{A}_k$ represent symbolically, with the right-shift operator $\hat{v}$, respectively the discrete signals $\varphi_p(x)$, $p = \overline{1, k}$ appearing in expression (3).

As the here proposed symbolic representation is a homomorphism to addition, a certain element in algebra $\mathcal{A}_k$:

$$\hat{w} = b_1 \hat{v}^0 + b_2 \hat{v} + \dots + b_k \hat{v}^{k-1}; \quad b_i \in \mathbb{R}, \quad i = \overline{1, k}$$

represents symbolically, with the right-shift operator $\hat{v}$, the discrete signal

$$f(n) = b_1 \varphi_1(n) + b_2 \varphi_2(n) + \dots + b_k \varphi_k(n),$$

i.e. any solution to the difference equation (4).

This way, it results

Proposition 2. *In order to represent symbolically discrete signals solutions to homogeneous difference equations with constant coefficients, polynomial algebras are sufficient.*

Therefore, any solution to a homogeneous difference equation with constant coefficients is representable symbolically by an element of the polynomial

algebra generated by the characteristic polynomial of the equation. The element generating the cyclical basis in algebra is its right-shift operator.

4. Let be

$$(13) \quad y_{n+k} + a_1 y_{n+k-1} + \dots + a_l y_n = f_n$$

a non-homogeneous difference equation with constant coefficients. A physical system functioning according to equation (13) is denoted as digital filter [5], [6].

The general solution to equation (13) represents the filter response to excitation f_n and has the form

$$y_n = y_n^* + y_n^{**},$$

where y_n^* is the general solution to the corresponding homogeneous equation and y_n^{**} a particular solution to equation (13).

As it has been shown, y_n^* is representable symbolically by an element belonging to the polynomial algebra $\mathcal{A}_k$ (2), generated by the characteristic polynomial of the homogeneous equation corresponding to equation (13):

$$\widehat{W}_1 = c_1 \widehat{v}^0 + c_2 \widehat{v}^1 + \dots + c_k \widehat{v}^{k-1}.$$

The set of constants $c_i, i = \overline{1, k}$ specifies certain initial conditions imposed to the digital filtering (13).

If the discrete signal f_n is representable symbolically on a finite order associative and commutative algebra over the real number field $\mathcal{A}_p$, by element $\widehat{w}$ with the right-shift operator $\widehat{D}$, a solution to equation (13) representable symbolically on algebra $\mathcal{A}_p$ may be determined.

Let $\widehat{W}_2$ be the element representing symbolically this solution. It shall be denoted as

$$\widehat{W}_2 \rightarrow y_n^{**}.$$

Taking into account the properties of representation [2]:

$$(14) \quad \widehat{D}^p \widehat{W}_2 \rightarrow y_{n+p}^{**}, \quad p = \overline{0, k}$$

From (13) and (14) it results that $\widehat{w}_2$ satisfies relation

$$(15) \quad \widehat{W}_2 [\widehat{D}^k + a_1 \widehat{D}^{k-1} + \dots + a_k \widehat{D}^0] = \widehat{W}.$$

If the characteristic polynomial of equation (13) is not an annihilator one for element $\widehat{D}$ and $p(\widehat{D})$ is not a divisor of zero in algebra $\mathcal{A}_p$, from (15) it results

$$(16) \quad \widehat{W}_2 = \widehat{W} p(\widehat{D})^{-1} = d_1 \widehat{e}_1 + d_2 \widehat{e}_2 + \dots + d_p \widehat{e}_p; \quad d_i \in \mathbb{R}, \quad i = \overline{1, p},$$

where $\{\widehat{e}_1, \widehat{e}_2, \dots, \widehat{e}_p\}$ is a basis in $\mathcal{A}_p$. If basis elements: $\widehat{e}_i, i = \overline{1, p}$ represent symbolically with the right-shift operator $\widehat{D}$, the discrete signals $f_i(n), i = \overline{1, p}$, it results

$$(17) \quad y_n^{**} = d_1 f_1(n) + d_2 f_2(n) + \dots + d_p f_p(n).$$

This way we have demonstrated

Proposition 3. *The digital filter (13) response is a discrete signal representable symbolically on the direct sum algebra $\mathcal{A}_k \oplus \mathcal{A}_p$ by element*

$$\widehat{W} = \widehat{W}_1 E_1 + \widehat{W}_2 E_2$$

with the right-shift operator

$$\widehat{D} = \widehat{v} E_1 + \widehat{D} E_2,$$

where E_1 and E_2 are two orthogonal idempotents in algebra $\mathcal{A}_k \oplus \mathcal{A}_p$.

Remark 1. With $p(\widehat{D})=0$ or $p(\widehat{D})$ as divisors of zero in algebra $\mathcal{A}_p$, a solution y_n^{**} representable symbolically on $\mathcal{A}_p$ but having another right-shift operator may be determined. This operator generally belongs to a subalgebra in algebra $\mathcal{A}_p$.

2. If algebras $\mathcal{A}_k$ and $\mathcal{A}_p$ are isomorphic and elements $\widehat{v} \in \mathcal{A}_k$ and $\widehat{D} \in \mathcal{A}_p$, which are right-shift operators correspond one to each other, the response of the digital filter (13) is representable symbolically on any of those algebras. This aspect characterizes resonance phenomenon.

3. Z-transform algebrizing certain difference equations [5], [6], is a classical method to determine the response of the digital filter. Under specified conditions, Z-transform associates to a discrete signal a complex function of a complex variable.

The present paper demonstrates that in order to determine this response, an efficient algorithmic proceeding may be used under certain circumstances. The described proceeding is an advantageous one because it does not require to determine the characteristic roots of the difference equation (13)¹⁾. This is a consequence of the more general fact that in certain conditions to a discrete, respectively „continual“, signal hypercomplex functions of a hypercomplex variable may be conveniently attached by extension of the frequently used Z-transform, Fourier transform [3].

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¹⁾ A similar result was obtained for continual signals by using another approach [1].

O CLASĂ DE FUNCȚII MULTIVOCE DE TIP EXPONENȚIAL DEFINITĂ PE MULȚIMEA NUMERELOR RAȚIONALE

DE

M. GONTINEAC

În [3] am pus în evidență o clasă de funcții multivoce definite pe mulțimea $\mathbb{Q}$ a numerelor raționale și luând valori în $\mathbb{R}$. Unui număr rațional nh îi corespunde prin funcția multivocă un șir de numere reale a_{nh}^k , k indicând ordinea termenilor acestui șir și unde avem $n \in \mathbb{Z}$, $k \in \mathbb{N}$ și $h = (k!)^{-1}$. În această clasă de funcții multivoce am definit o clasă de operatori liniari numiți convențional operatori de derivare pentru care am demonstrat o serie de proprietăți. Amintim două dintre aceste definiții.

Definiția 1. Numim derivata la dreapta a lui a_{nh}^k în modul q cu $q \in \mathbb{N}$, operatorul notat $\Delta_q a_{nh}^k$ definit prin

$$(1) \quad \Delta_q a_{nh}^k = \sum_{i=0}^q \alpha_i a_{(n+i)h}^k,$$

unde coeficienții $\alpha_0, \alpha_1, \dots, \alpha_q$ sînt fixați în $\mathbb{R}$ și satisfac condițiile

$$(1.1) \quad \sum_{i=0}^q \alpha_i = 0,$$

$$(1.2) \quad \sum_{i=0}^q i \alpha_i = \frac{1}{h}.$$

Definiția 2. Numim operatori de derivare bază a lui a_{nh}^k după modul m , operatorii dați de

$$(2) \quad d_m a_{nh}^k = \frac{a_{(n+m)h}^k - a_{nh}^k}{mh},$$

unde $m \in \mathbb{Z} - \{0\}$.

Tot în [3] am determinat termenii șirului inițial cunoscînd termenii șirului căruia i-am aplicat operatorii de derivare bază (2). Astfel am obținut

$$a_{(sm+p)h}^k = mh \sum_{i=0}^{s-1} d_m a_{(im+p)h}^k + a_{ph}^k \quad \text{cu } s \in \mathbb{N} \quad \text{și } p=0, 1, \dots, m-1$$

și observăm că șirul inițial depinde de m constante arbitrare a_{ph}^k cu $p = 0, 1, \dots, m-1$.

În această lucrare vom pune în evidență o clasă de funcții multivoce definite pe mulțimea numerelor raționale pozitive, fiecare dintre aceste funcții fiind un analog al funcției exponențiale, accentul punându-se pe calculul șirului de valori în orice număr rațional pozitiv.

Vom determina mai întâi termenii șirului inițial cunoscând termenii șirului căruia i-am aplicat operatorii de derivare (1) pentru cazul $q=2$. Din (1), pentru $q=2$ rezultă

$$(3) \quad \Delta_2 a_{nh}^k = \sum_{i=0}^2 \alpha_i a_{(n+i)h}^k,$$

cu

$$(3.1) \quad \sum_{i=0}^2 \alpha_i = 0 \quad \text{și} \quad (3.2) \quad \sum_{i=0}^2 i \alpha_i = \frac{1}{h}.$$

Cunoscute fiind $\Delta_2 a_{nh}^k$, $\Delta_2 a_{(n+1)h}^k$, $\Delta_2 a_{(n+2)h}^k$ să determinăm în funcție de constante arbitrare a_{nh}^k , $a_{(n+1)h}^k$ termenii $a_{(n+2)h}^k$, $a_{(n+3)h}^k$, $a_{(n+4)h}^k$. Problema revine la rezolvarea sistemului

$$(4) \quad \begin{aligned} \Delta_2 a_{nh}^k &= \alpha_0 a_{nh}^k + \alpha_1 a_{(n+1)h}^k + \alpha_2 a_{(n+2)h}^k, \\ \Delta_2 a_{(n+1)h}^k &= \alpha_0 a_{(n+1)h}^k + \alpha_1 a_{(n+2)h}^k + \alpha_2 a_{(n+3)h}^k, \\ \Delta_2 a_{(n+2)h}^k &= \alpha_0 a_{(n+2)h}^k + \alpha_1 a_{(n+3)h}^k + \alpha_2 a_{(n+4)h}^k, \end{aligned}$$

în necunoscutele $a_{(n+2)h}^k$, $a_{(n+3)h}^k$, $a_{(n+4)h}^k$. Folosind condițiile (3.1) și (3.2), sistemul (4) devine

$$(5) \quad \begin{pmatrix} h\alpha_2 & 0 & 0 \\ 1 & h\alpha_2 & 0 \\ \text{cu } 1 & 1 & h\alpha_2 \end{pmatrix} \begin{pmatrix} a_{(n+2)h}^k \\ a_{(n+3)h}^k \\ a_{(n+4)h}^k \end{pmatrix} = \begin{pmatrix} h\Delta_2 a_{nh}^k - h\alpha_0 a_{nh}^k - h\alpha_1 a_{(n+1)h}^k \\ h\Delta_2 a_{(n+1)h}^k - 2h\Delta_2 a_{nh}^k - 2h\alpha_0 a_{nh}^k - h(\alpha_0 + 2\alpha_1) a_{(n+1)h}^k \\ h\Delta_2 a_{(n+2)h}^k - 2h\Delta_2 a_{(n+1)h}^k + 3h\Delta_2 a_{nh}^k - 3h\alpha_0 a_{nh}^k - h(2\alpha_0 + 3\alpha_1) a_{(n+1)h}^k \end{pmatrix}$$

$$\det \begin{pmatrix} h\alpha_2 & 0 & 0 \\ 1 & h\alpha_2 & 0 \\ 1 & 1 & h\alpha_2 \end{pmatrix} = (h\alpha_2)^3 \neq 0,$$

deoarece dacă pentru $\alpha_2=0$ avem $\alpha_1=1/h$ și obținem un mod de derivare bază tratat în [3].

Din (5) obținem

$$\begin{pmatrix} a_{(n+2)h}^k \\ a_{(n+3)h}^k \\ a_{(n+4)h}^k \end{pmatrix} = \frac{1}{(h\alpha_2)^3} \begin{pmatrix} (h\alpha_2)^2 & 0 & 0 \\ -h\alpha_2 & (h\alpha_2)^2 & 0 \\ 1-h\alpha_2-h\alpha_2(h\alpha_2)^2 \end{pmatrix} \begin{pmatrix} h\Delta_2 a_{nh}^k - h\alpha_0 a_{nh}^k - h\alpha_1 a_{(n+1)h}^k \\ h\Delta_2 a_{(n+1)h}^k - 2h\Delta_2 a_{nh}^k - 2\alpha_0 a_{nh}^k - h(\alpha_0 + 2\alpha_1) a_{(n+1)h}^k \\ h\Delta_2 a_{(n+2)h}^k - 2h\Delta_2 a_{(n+1)h}^k + 3h\Delta_2 a_{nh}^k - 3h\alpha_0 a_{nh}^k - h(2\alpha_0 + 3\alpha_1) a_{(n+1)h}^k \end{pmatrix} \quad (6)$$

Observația 1. Cazul $q>2$ nu prezintă decît dificultăți calculatorii.

În continuare ne punem problema determinării funcției exponențiale multivoce cu obținerea de diferite ramuri și cu un număr diferit de constante arbitrare după modul de derivare.

Se știe că funcția exponențială e^x se poate defini ca soluție a ecuației diferențiale

$$(6) \quad y' - y = 0 \quad \text{cu condiția} \quad (6.1) \quad y(0) = 1.$$

În (2) se consideră modul $m=1$ de derivare bază și pasul $h=(k!)^{-1}$ și atunci

$$\alpha_1 a_n^{\frac{k}{k!}} = k! \left(a_{n+1}^{\frac{k}{k!}} - a_n^{\frac{k}{k!}} \right).$$

Atașăm lui y pe $a_n^{\frac{k}{k!}}$ iar lui y' pe $d_1 a_n^{\frac{k}{k!}}$ și atunci (6) devine

$$(7) \quad a_{n+1}^{\frac{k}{k!}} - \frac{k!+1}{k!} a_n^{\frac{k}{k!}} = 0.$$

Notăm $a_n^{\frac{k}{k!}} = b_n$ și deci $a_{n+1}^{\frac{k}{k!}} = b_{n+1}$ iar (7) se scrie sub forma

$$(8) \quad b_{n+1} - \frac{k!+1}{k!} b_n = 0,$$

care este o ecuație cu diferențe rezolvabilă după procedeul clasic.

Căutăm soluții de forma $b_n = \lambda^n$ și atunci (8) devine

$$\lambda - \frac{k!+1}{k!} = 0 \quad \text{de unde} \quad b_n = C^k \left(\frac{k!+1}{k!} \right)^n$$

și deci

$$(9) \quad a_n^{\frac{k}{k!}} = C^k \left(\frac{k!+1}{k!} \right)^n.$$

Observația 2. Determinarea soluției ecuației (7) revine la a căuta soluții de forma $a_n^{\frac{k}{k!}} = \lambda^n$ iar $a_{n+1}^{\frac{k}{k!}} = \lambda^{n+1}$ pentru (7).

Pentru determinarea exponențialei multivoce corespunzătoare modului $m=1$ de derivare impunem condiția $a_0^k=1$ de forma (6.1) de unde

$$a_n^{\frac{k}{k!}} = \left(\frac{k!+1}{k!} \right)^n.$$

Observația 3. Valorile exponențialei multivoce în 1 sînt $a_1^1, a_1^2, \dots, \dots, a_1^k, \dots$ unde $a_1^k = \left(\frac{k!+1}{k!} \right)^{k!}$.

Pentru pasul h suficient de mic rezultă $\lim_{k \rightarrow \infty} a_1^k = e$.

În (2) considerăm modul $m=2$ și $h=(k!)^{-1}$. Atunci

$$d_2 a_n^{\frac{k}{k!}} = \frac{k!}{2} \left(a_{n+2}^{\frac{k}{k!}} - a_n^{\frac{k}{k!}} \right).$$

Atașăm lui y pe $a_n^{\frac{k}{k!}}$ și lui y' pe $d_2 a_n^{\frac{k}{k!}}$ și atunci ecuația (6) se scrie

$$(10) \quad a_{n+2}^{\frac{k}{k!}} - \frac{k!+2}{k!} a_n^{\frac{k}{k!}} = 0.$$

Ținând seama de observația 2 căutăm soluții de forma $a_n^k = \lambda^n$ iar $a_{n+2}^k = \lambda^{n+2}$ și din (10) obținem $\lambda_{1,2} = \pm \sqrt{(k!+2)/k!}$ și deci

$$(11) \quad a_{\frac{k}{k!}}^n = \left(\frac{k!+2}{k!} \right)^{\frac{n}{2}} (C_1^k + (-1)^n C_2^k).$$

Pentru determinarea constantelor C_1^k și C_2^k impunem condițiile

$$(10.1) \quad a_0^k = 1, \quad (10.2) \quad a_{\frac{1}{k!}}^k = 1$$

în (11), de unde

$$C_1^k = \frac{\sqrt{\frac{k!+2}{k!}} + 1}{2\sqrt{\frac{k!+2}{k!}}}, \quad C_2^k = \frac{\sqrt{\frac{k!+2}{k!}} - 1}{2\sqrt{\frac{k!+2}{k!}}}.$$

Atunci (11) se scrie

$$(12) \quad a_{\frac{k}{k!}}^n = \left(\frac{k!+2}{k!} \right)^{\frac{n}{2}} \frac{\sqrt{\frac{k!+2}{k!}} + 1 + (-1)^n \left(\sqrt{\frac{k!+2}{k!}} - 1 \right)}{2\sqrt{\frac{k!+2}{k!}}}.$$

Observația 4. Valorile exponențialei multivoce într-un număr rațional pozitiv $p(q!)^{-1}$ fixat vor fi $a_{\frac{p}{q!}}^k$ cu $k \geq q$ și se obține astfel:

a) pentru $k=q$, $n=p$:

$$a_{\frac{p}{q!}}^q = \begin{cases} \left(\frac{q!+2}{q!} \right)^{\frac{p}{2}} & \text{dacă } p=2l, \\ \left(\frac{q!+2}{q!} \right)^{\frac{p-1}{2}} & \text{dacă } p=2l-1; \end{cases}$$

b) pentru $k=q+1$, $n=p(q+1)$:

$$a_{\frac{p}{q!}}^{q+1} = \begin{cases} \left(\frac{(q+1)!+2}{(q+1)!} \right)^{\frac{p(q+1)}{2}} & \text{dacă } p(q+1)=2l, \\ \left(\frac{(q+1)!+2}{(q+1)!} \right)^{\frac{p(q+1)-1}{2}} & \text{dacă } p(q+1)=2l-1; \end{cases}$$

c) pentru $k \geq q+2$ și $n=p(q+1)(q+2) \cdots k$ corespunzător

$$a_{\frac{p}{q!}}^k = \left(\frac{k!+2}{k!} \right)^{\frac{p(q+1)(q+2) \cdots k}{2}};$$

pentru h suficient de mic rezultă $\lim_{k \rightarrow \infty} a_{\frac{p}{q!}}^k = e^{\frac{p}{q!}}$.

În (2) considerăm $m=3$ și $h=(k!)^{-1}$ și atunci

$$d_3 a_n^k = \frac{k!}{3} \left(a_{n+3}^k - a_n^k \right).$$

Ecuția (6) devine

$$(13) \quad a_{n+3}^k - \frac{k!+3}{k!} a_n^k = 0.$$

Procedînd ca în cazurile anterioare obținem soluția ecuației (13) sub forma

$$(14) \quad a_n^k = \left(\frac{k!+3}{k!} \right)^{\frac{n}{3}} \left(C_1^k + C_2^k \cos \frac{2n\pi}{3} + C_3^k \sin \frac{2n\pi}{3} \right).$$

Pentru determinarea constantelor C_1^k , C_2^k , C_3^k impunem în relația (14) condițiile

$$(13.1) \quad a_0^k = 1, \quad (13.2) \quad a_{\frac{1}{k}}^k = \alpha, \quad (13.3) \quad a_{\frac{2}{k}}^k = \beta,$$

cu α , β oarecare dar fixați în $\mathbb{R}$, de unde

$$(15) \quad \begin{aligned} C_1^k &= -1 + \left(\frac{k!+3}{k!} \right)^{-\frac{1}{3}} \alpha + \left(\frac{k!+3}{k!} \right)^{-\frac{2}{3}} \beta, \\ C_2^k &= 2 - \left(\frac{k!+3}{k!} \right)^{-\frac{1}{3}} \alpha - \left(\frac{k!+3}{k!} \right)^{-\frac{2}{3}} \beta, \\ C_3^k &= \frac{1}{\sqrt{3}} \left[\left(\frac{k!+3}{k!} \right)^{-\frac{1}{3}} \alpha + \left(\frac{k!+3}{k!} \right)^{-\frac{2}{3}} \beta \right]. \end{aligned}$$

Observația 5. Valorile exponențialei multivoce într-un număr rațional pozitiv fixat $p(q!)^{-1}$ vor fi $a_{\frac{p}{q!}}^k$ cu $k \geq q$ unde

$$a_{\frac{p}{q!}}^k = \left(\frac{k!+3}{k!} \right)^{\frac{p(q+1)\dots k}{3}} \left(C_1^k + C_2^k \cos \frac{2p(q+1)\dots k}{3} + C_3^k \sin \frac{2p(q+1)\dots k}{3} \right),$$

cu C_1^k , C_2^k , C_3^k determinați în (15).

Pentru $k \geq q+3$, ținînd seama de $C_1^k + C_2^k = 1$ din (15), obținem

$$a_{\frac{p}{q!}}^k = \left(\frac{k!+3}{k!} \right)^{\frac{p(q+1)\dots k}{3}}.$$

Pentru h suficient de mic rezultă $\lim_{h \rightarrow \infty} a_{\frac{p}{q!}}^h = e^{\frac{p}{q!}}$.

În (1) considerăm modul $q=2$ de derivare a lui a_{nh}^k și atunci

$$(16) \quad \Delta_2 a_{nh}^k = \sum_{i=0}^2 \alpha_i a_{(n+i)h}^k,$$

coeficienții α_i fixați în $\mathbb{R}$ satisfăcînd condițiile

$$(16.1) \quad \sum_{i=1}^2 \alpha_i = 0, \quad (16.2) \quad \sum_{i=0}^2 i \alpha_i = \frac{1}{h}.$$

Pentru fixarea coeficienților α_i mai impunem condiția

$$(16.3) \quad \alpha_i + 4\alpha_2 = \frac{\alpha}{h}$$

pentru orice α fixat în $\mathbb{R}$ și atunci

$$\alpha_1 = \frac{2-\alpha}{h}, \quad \alpha_2 = \frac{\alpha-1}{2h}, \quad \alpha_0 = \frac{\alpha-3}{2h}$$

și deci (16) devine

$$(17) \quad \Delta_2 a_{nh}^k = \frac{\alpha-3}{2h} a_{nh}^k + \frac{2-\alpha}{h} a_{(n+1)h}^k + \frac{\alpha-1}{2h} a_{(n+2)h}^k.$$

Ecuția (6) devine

$$(18) \quad (\alpha-1) a_{(n+2)h}^k - 2(\alpha-2) a_{(n+1)h}^k + (\alpha-3-2h) a_{nh}^k = 0.$$

În (18) căutăm soluții de forma $a_{nh}^k = \lambda^n$ și obținem pe λ ca soluție a ecuației

$$(19) \quad (\alpha-1)\lambda^2 - 2(\alpha-2)\lambda + (\alpha-3-2h) = 0.$$

Notăm discriminantul ecuației (19) cu $\delta = 1 + 2h(\alpha-1)$ și deoarece α este fixat putem lua h astfel încât $\delta > 0$ și deci soluțiile ecuației (19) vor fi $\lambda_1, \lambda_2 \in \mathbb{R}$ cu $\lambda_1 \neq \lambda_2$; avem

$$\lambda_{1,2} = \frac{\alpha-2 \pm \sqrt{\delta}}{\alpha-1},$$

cu $\alpha \neq 1$ deoarece dacă $\alpha=1$ se obțin în (16) operatori de derivare bază.

Atunci soluția ecuației (18) este

$$(20) \quad a_{nh}^k = C_1^k \left(\frac{\alpha-2+\sqrt{\delta}}{\alpha-1} \right)^n + C_2^k \left(\frac{\alpha-2-\sqrt{\delta}}{\alpha-1} \right)^n.$$

Pentru $h = (k!)^{-1}$ relația (20) devine

$$(21) \quad a_{\frac{n}{k}}^k = C_1^k \left[\frac{\alpha-2+\sqrt{\frac{k!+2(\alpha-1)}{k!}}}{\alpha-1} \right]^n + C_2^k \left[\frac{\alpha-2-\sqrt{\frac{k!+2(\alpha-1)}{k!}}}{\alpha-1} \right]^n.$$

Pentru determinarea constantelor C_1^k, C_2^k impunem în (21) condițiile $a_0^k = 1, a_{\frac{k}{k}}^k = 1$ și obținem

$$(22) \quad C_1^k = \frac{1 + \sqrt{\frac{k!+2(\alpha-1)}{k!}}}{2\sqrt{\frac{k!+2(\alpha-1)}{k!}}}, \quad C_2^k = \frac{\sqrt{\frac{k!+2(\alpha-1)}{k!}} - 1}{2\sqrt{\frac{k!+2(\alpha-1)}{k!}}}.$$

Observația 6. Valorile exponențialei multivoce într-un număr rațional pozitiv fixat $f(q!)^{-1}$ sînt $a_{q!}^k$ cu $k \geq q$ unde

$$a_{q!}^k = C_1^k \left[\frac{\alpha - 2 + \sqrt{\frac{k! + 2(\alpha - 1)}{k!}}}{\alpha - 1} \right]^{p(q+1) \dots k} + C_2^k \left[\frac{\alpha - 2 - \sqrt{\frac{k! + 2(\alpha - 1)}{k!}}}{\alpha - 1} \right]^{p(q+1) \dots k}$$

cu C_1^k și C_2^k determinați în (22).

Este suficient să luăm $\alpha > 2$ pentru ca $\lim_{k \rightarrow \infty} a_{q!}^k = e^{\frac{p}{q!}}$.

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A CLASS OF MULTIVALUED FUNCTIONS OF EXPONENTIAL TYPE, DEFINED ON RATIONAL NUMBERS

(Summary)

in this Note we study a class of multivalued functions, each of them being an analogon of the exponential.

A FAMILY OF MULTISTEP METHODS FOR SECOND-ORDER DIFFERENTIAL EQUATIONS

BY

N. CALISTRU

1. Introduction. In [2], T. C. Zverkina established and studied a certain family of multistep methods for numerical solution of Cauchy's problem for first-order differential systems.

In this paper, we shall use the same method for the numerical solution of the problem

$$(1.1) \quad \ddot{x} = f(t, x, \dot{x}), \quad t \in \mathbb{R},$$

$$(1.2) \quad x(t_0) = x_0, \quad \dot{x}(t_0) = \dot{x}_0, \quad t_0 \in \mathbb{R}.$$

Here, $x = x(t) = (x^1, x^2, \dots, x^n)^T$ is the unknown vector function, $x_0 = (x_0^1, x_0^2, \dots, x_0^n)^T$ and $\dot{x}_0 = (\dot{x}_0^1, \dot{x}_0^2, \dots, \dot{x}_0^n)^T$ are prescribed vectors and $f(t, z, w) = (f^1, f^2, \dots, f^n)^T$ is a vector function defined on $J \times \mathbb{R}^{2n} \rightarrow \mathbb{R}^n$, $J \subset \mathbb{R}$.

Remark 1.1. Usually, the methods of solution for first-order differential systems, can be applied for solving the systems of high-order differential equations. This fact does not diminish the importance of determining some special methods for the solution of high-order differential equations. Moreover, it is proved (see e.g. [1]) that such methods are preferable to the first methods.

2. Preliminaries. In the following we use the notations: $\mathbb{Z}$, the set of integer numbers; $\mathbb{R}$, the real line; J an interval of the real line: $\mathbb{R}^n = \{x/x = (x^1, x^2, \dots, x^n)^T, x^i \in \mathbb{R}, i = \overline{1, n}\}$ with the norm $\|x\| = \sup_{i=1, n} |x^i|$.

We shall suppose that the problem (1.1), (1.2) has a unique continuous and sufficiently smooth solution on J . We denote with $x(t)$ this solution.

Let $T \in J$, $T > t_0$ be a fixed constant and $h \in (0, T - t_0)$. We shall consider the following sets of points from $[t_0, T]$

$$(2.1) \quad \Sigma_h \equiv \{t_j/t_j = t_0 + jh, j = 0, N_1\}, \quad N_1 = [(T - t_0)/h],$$

$$(2.2) \quad \Sigma_h(\alpha) \equiv \{t_{j-\alpha}/t_{j-\alpha} \equiv t_0 + (j - \alpha)h, j = -[\alpha], -[\alpha] + 1, \dots, [\alpha] + (T - t_0)/h\},$$

where α is a real parameter, $[a]$ is the integer part of the real number a . We shall call Σ_h and $\Sigma_h(\alpha)$ a principal and auxiliary net of knots on $[t_0, T]$.

Remark 2.1. One can easily verify that $\Sigma_h(\alpha) \equiv \Sigma_h$ for $\alpha \in \mathbb{Z}$. In what follows, we shall use the interpolation of vector functions and the next obvious assertion: for any continuous and twice continuous differentiable vector function $F: J \rightarrow \mathbb{R}^n$ we have

$$(2.3) \quad \dot{F}(\bar{t}+s) - \dot{F}(\bar{t}) = \int_{\bar{t}}^{\bar{t}+s} \ddot{F}(t) dt,$$

$$(2.4) \quad F(\bar{t}+s) - F(\bar{t}) = \int_{\bar{t}}^{\bar{t}+s} (\bar{t}+s-t) F(t) dt + s\dot{F}(\bar{t}),$$

for every $\bar{t} \in J$, $s \in \mathbb{R}$ with $\bar{t}+s \in J$.

3. Construction of Family of Multistep Methods. a) *The principal family of multistep methods.* Let $x(t)$ be the solution of (1.1), (1.2), $\{t_{k-j-\alpha}\}$, $j=0, \bar{p}$, $k \geq 1$, $p \geq 1$, $k, p \in N$ be $p+1$ knots from $\Sigma_h(\alpha)$ and let $x(t_{k-j-\alpha})$, $\dot{x}(t_{k-j-\alpha})$ be the accurate values of functions $x(t)$, $\dot{x}(t)$ in these knots. ($\dot{x}$ is the derivative of the function x).

From

$$(3.1) \quad \ddot{x}(t) = f(t, x(t), \dot{x}(t)),$$

we get the values $\ddot{x}(t_{k-j-\alpha})$ $j=0, \bar{p}$, and using Newton's interpolation formula with backward differences, the function $\ddot{x}(t)$ can be represented in the form

$$(3.2) \quad \ddot{x}(t) = X_p(\tau) + R_p(\tau, \ddot{x}),$$

where

$$(3.3) \quad X_p(\tau) = \sum_{i=0}^p \frac{(\tau)_i}{i!} \nabla^i \ddot{x}(t_{k-\alpha})$$

is the interpolating polynomial and

$$(3.4) \quad R_p(\tau, \ddot{x}) = \frac{(\tau)_{p+1}}{(p+1)!} h^{p+1} x^{(p+3)} + \mathcal{O}(h^{p+2}),$$

is the remainder term of the interpolation formula. (We have utilised the notation

$$t = t_{k-\alpha} + \tau h; \quad (\tau)_i = \tau(\tau+1) \dots (\tau+i-1), \quad (\tau)_0 \equiv 1;$$

$$\nabla^i \ddot{x}(t_j) = \nabla^{i-1} \ddot{x}(t_j) - \nabla^{i-1} \ddot{x}(t_{j-1}), \quad \nabla^0 x(t) = x(t), \quad i=1, 2, \dots).$$

Considering (2.3) and (2.4) for $\bar{t} = t_{k-1} \in \Sigma_h$, $s = h$, $\ddot{F}(t) = \ddot{x}(t)$ and taking into account (3.3), (3.4) we get

$$(3.5) \quad \dot{x}(t_k) = \dot{x}(t_{k-1}) + h \sum_{i=0}^p P_i(\alpha) \nabla^i \ddot{x}(t_{k-\alpha}) + \rho_{p,k}(\alpha),$$

$$(3.6) \quad x(t_k) = x(t_{k-1}) + h \dot{x}(t_{k-1}) + h^2 \sum_{i=0}^p Q_i(\alpha) \nabla^i \ddot{x}(t_{k-\alpha}) + \tau_{p,k}(\alpha),$$

where

$$(3.7) \quad P_i(\alpha) = \frac{1}{i!} \int_{\alpha-1}^{\alpha} (\tau)_i d\tau, \quad Q_i(\alpha) = \frac{1}{i!} \int_{\alpha-1}^{\alpha} (\alpha-\tau) (\tau)_i d\tau, \quad i=0, 1, \dots$$

$$(3.8) \quad \rho_{p,k}(\alpha) = h^{p+2} P_{p+1}(\alpha) \chi^{(p+3)}(t_{k-\alpha}) + O(h^{p+3}),$$

$$(3.9) \quad \tau_{p,k}(\alpha) = h^{p+2} Q_{p+1}(\alpha) \chi^{(p+4)}(t_{k-\alpha}) + O(h^{p+4}).$$

Let $u, \dot{u}, \ddot{u}$ be real n -dimensional vector functions defined on $\sum_h \cup \sum_h(\alpha)$ such that

$$(3.10) \quad \ddot{u}_{j-\alpha} = f(t_{j-\alpha}, u_{j-\alpha}, \dot{u}_{j-\alpha}), \quad j = -p, -p+1, \dots$$

$$(A) \quad \begin{cases} \dot{u}_k(\alpha) = \dot{u}_{k-1}(\alpha) + h \sum_{i=0}^p P_i(\alpha) \nabla^i \ddot{u}_{k-\alpha}(\alpha), \\ u_k(\alpha) = u_{k-1}(\alpha) + h \dot{u}_{k-1}(\alpha) + h^2 \sum_{i=0}^p Q_i(\alpha) \nabla^i \ddot{u}_{k-\alpha}(\alpha), \quad k = \overline{1, N_1}, \end{cases}$$

where w_i means the value of the vector function w for $t=t_i$.

We denote by $\mathcal{M}_A^{(k)}(\alpha)$ the set

$$\mathcal{M}_A^{(k)}(\alpha) = \{u_{i-1}(\alpha), \dot{u}_{i-1}(\alpha), u_{k-j-\alpha}(\alpha), \dot{u}_{k-j-\alpha}(\alpha); j = \overline{0, p}; k \geq 1\},$$

and by (αA) the triplet

$$(\alpha A) = \{\mathcal{M}_A^{(k)}(\alpha), (3.10), (A)\}.$$

We note that if the set $\mathcal{M}_A^{(k)}(\alpha)$ is known for certain $\alpha \in \mathbb{R}$ then the algorithm (αA) permits to find $u_k, \dot{u}_k$ for $t=t_k \in \sum_h$. Therefore, whenever the values of the functions $u, \dot{u}$ on $\sum_h(\alpha) \cup \{t_0\}$ are determined, then by using the corresponding method (αA) their values can be calculated in all the knots of the principal network. In this sense we say that (αA) is a principal family of multistep methods. It is clear that neither method of the principal family can be used unless it is a priori known a way of finding $\mathcal{M}_A^{(k)}(\alpha)$.

b) *The auxiliary family of multistep methods.* We have seen above that for the use of the principal family methods, it is necessary to know the values of the involved functions on the auxiliary network $\sum_h(\alpha)$.

These values, except some of them, can be obtained by using a method belonging to the family

$$(\alpha A^*) = \{\mathcal{M}_{A^*}^{(k)}(\alpha), (3.10), (A^*)\},$$

where

$$(A^*) \begin{cases} \dot{u}_{k-\alpha}(\alpha) = \dot{u}_{k-1}(\alpha) + h \sum_{i=0}^p P_i^*(\alpha) \nabla^i \ddot{u}_{k-1-\alpha}(\alpha), \\ u_{k-\alpha}(\alpha) = u_{k-1}(\alpha) + h(1-\alpha) \dot{u}_{k-1}(\alpha) + h^2 \sum_{i=0}^p Q_i^*(\alpha) \nabla^i \ddot{u}_{k-1-\alpha}(\alpha), \quad k=1, \overline{N_1} \end{cases}$$

with

$$(3.11) \quad P_i^*(\alpha) = \frac{1}{i!} \int_{\alpha-1}^0 (\tau+1)_i d\tau, \quad Q_i^*(\alpha) = \frac{1}{i!} \int_0^{\alpha-1} (\tau)_{i+1} d\tau, \quad i=0, 1, 2, \dots$$

$$\mathcal{M}_{A^*}^{(k)}(\alpha) = \{\dot{u}_{k-1}(\alpha), \ddot{u}_{k-1}(\alpha), \ddot{u}_{k-j-\alpha}(\alpha), \ddot{u}_{k-j-\alpha}(\alpha) : j=0, \overline{p+1}\}.$$

Remark 3.1. The formulae (A^*) may be obtained by the same way as the formulae (A) . Namely, we use relations (2.3), (2.4) in which we take $\bar{t} = t_{k-1}$, $s = (1-\alpha)h$ and $\bar{F}$ is substituted by $\ddot{x}(t) = X_p^*(\tau) + R_p^*(\tau, \ddot{x})$.

Here, by $X_p^*(\tau)$ we denote the Newton's interpolating polynomial with respect to the knots $\{t_{k-j-\alpha}\}$, $j=0, \overline{p+1}$, for the function $\ddot{x}(t)$, and

$$(3.12) \quad R_p^*(\tau, \ddot{x}) = h^{p+1} \frac{(\tau+1)_{p+1}}{(p+1)!} x^{(p+3)}(t_{k-1-\alpha}) + \mathcal{O}(h^{p+2}),$$

is the remainder term. By this way, we obtain the following relations (which are similar to (3.5), (3.6)),

$$(3.13) \quad \dot{x}(t_{k-\alpha}) = \dot{x}(t_{k-1}) + h \sum_{i=0}^p P_i^*(\alpha) \nabla^i \ddot{x}(t_{k-1-\alpha}) + \rho_{p,k}^*(\alpha),$$

$$(3.14) \quad x(t_{k-\alpha}) = x(t_{k-1}) + h(1-\alpha) \dot{x}(t_{k-1}) + h^2 \sum_{i=0}^p Q_i^*(\alpha) \nabla^i \ddot{x}(t_{k-1-\alpha}) + \tau_{p,k}^*(\alpha),$$

where

$$(3.15) \quad \rho_{p,k}^*(\alpha) = h^{p+2} P_{p+1}^*(\alpha) x^{(p+3)}(t_{k-1-\alpha}) + \mathcal{O}(h^{p+3}),$$

$$(3.16) \quad \tau_{p,k}^*(\alpha) = h^{p+3} Q_{p+1}^*(\alpha) x^{(p+3)}(t_{k-1-\alpha}) + \mathcal{O}(h^{p+4}).$$

We call (αA^*) the auxiliary family of multistep methods.

Remark 3.2. For $\alpha \in \mathbb{R} \setminus \mathbb{Z}$, this family differs from the principal family. For $\alpha \in \mathbb{Z}$, both the auxiliary and principal family become usual multistep methods.

If we suppose that the set $\mathcal{M}_{A^*}^{(k)}(\alpha)$ is known for some $\alpha = \bar{\alpha} \in \mathbb{R} \setminus \mathbb{Z}$ then by the auxiliary method $(\bar{\alpha} A^*)$, we obtain the values $u_{k-\bar{\alpha}}(\bar{\alpha}), \ddot{u}_{k-\bar{\alpha}}(\bar{\alpha})$ which correspond to $t_{k-\bar{\alpha}} \in \sum_h(\bar{\alpha})$. Then by using the principal method $(\bar{\alpha} A)$ one can obtain the values $u_k(\bar{\alpha}), \dot{u}_k(\bar{\alpha})$ which correspond to $t_k \in \sum_h$.

Therefore, for a given $\alpha \in \mathbb{R} \setminus \mathbb{Z}$, we have to use two methods: one of them belonging to the principal family and the other one to the auxiliary family. Such a couple of methods of two families will be denoted by

$$(3.17) \quad (\alpha A A^*) = \{\mathcal{M}_{AA^*}^{(k)}(\alpha), (3.10), (A), (A^*)\},$$

where

$$\mathcal{M}_{AA^*}^{(k)}(\alpha) = \mathcal{M}_A^{(k)}(\alpha) \cup \mathcal{M}_{A^*}^{(k)}(\alpha).$$

The least value of k for which any algorithm of the family $(\alpha A A^*)$ may be applied is $k=1$. This involves the knowledge of the set $\mathcal{M}_{AA^*}^{(1)}(\alpha)$ in advance. For this reason the set $\mathcal{M}_{AA^*}^{(1)}(\alpha)$ is called the set of start values of the algorithms $(\alpha A A^*)$.

Remark 3.3. If the elements of the start set satisfy the conditions

$$u_0 = x(t_0), \quad \dot{u}_0 = \dot{x}(t_0), \quad u_{1-j-\alpha}(\alpha) \cong x(t_{1-j-\alpha}),$$

$$\dot{u}_{1-j-\alpha}(\alpha) \cong \dot{x}(t_{1-j-\alpha}), \quad j = \overline{0, p+1},$$

where $x(t)$ is the solution of the problem (1.1), (1.2), then we expect that the algorithms of the family $(\alpha A A^*)$ defined by (3.17) to be a family of algorithms with finite differences for numerical solution of the problem (1.1), (1.2).

Remark 3.4. For the using of the digital computers, it is suitable to express the algorithm of the family $(\alpha A A^*)$ without the operator ∇ . So, according to the relation

$$\nabla^m w_q = \sum_{j=0}^m \frac{(-m)_j}{j!} w_{q-j}$$

after some calculations, the formulae (A), (A^*) become

$$(A) \quad \begin{cases} \dot{u}_k = \dot{u}_{k-1} + h \sum_{i=0}^p A_i(\alpha) \ddot{u}_{k-i-\alpha}, \\ u_k = u_{k-1} + h \dot{u}_{k-1} + h^2 \sum_{i=0}^p \bar{A}_i(\alpha) \ddot{u}_{k-i-\alpha}, \end{cases}$$

$$(A^*) \quad \begin{cases} \dot{u}_{k-\alpha} = \dot{u}_{k-1} + h \sum_{i=0}^p A_i^*(\alpha) \ddot{u}_{k-i-1-\alpha}, \\ u_{k-\alpha} = u_{k-1} + h(1-\alpha) \dot{u}_{k-1} + h^2 \sum_{i=0}^p \bar{A}_i^*(\alpha) \ddot{u}_{k-i-1-\alpha}, \quad k = \overline{1, N_1}, \end{cases}$$

where

$$(3.18) \quad A_i(\alpha) = \sum_{j=i}^p \frac{(-i)_j}{j!} P_j(\alpha); \quad \bar{A}_i(\alpha) = \sum_{j=i}^p \frac{(-j)_i}{i!} Q_j(\alpha),$$

$$(3.19) \quad A_i^*(\alpha) = \sum_{j=i}^p \frac{(-j)_i}{i!} P_j^*(\alpha); \quad \bar{A}_i^*(\alpha) = \sum_{j=i}^p \frac{(-j)_i}{i!} Q_j^*(\alpha), \quad i = \overline{0, 1, 2, \dots}$$

Remark 3.5. By virtue of the relations (3.7), (3.11) all the coefficients of the family $(\alpha A A^*)$ are algebraic polynomials of the parameter α .

Although the specified formulae are simple, the calculation of the coefficients by them is not so easy. Fortunately, these coefficients satisfy certain recurrence relations.

Lemma 3.1. *The polynomials $P_n(\alpha)$, $Q_n(\alpha)$, $P_n^*(\alpha)$ and $Q_n^*(\alpha)$ verify the following recurrence relations;*

$$(3.20) \quad P_n(\alpha) + \frac{1}{2} P_{n-1}(\alpha) + \cdots + \frac{1}{n+1} P_0(\alpha) = \frac{(\alpha)_n}{n!},$$

$$(3.21) \quad Q_n(\alpha) + \frac{1}{2} Q_{n-1}(\alpha) + \cdots + \frac{1}{n+1} Q_0(\alpha) = \Gamma_{n+1}(\alpha) - \frac{(\alpha-1)_{n+1}}{(n+1)!},$$

$$(3.22) \quad P_n^*(\alpha) + \frac{1}{2} P_{n-1}^*(\alpha) + \cdots + \frac{1}{n+1} P_0^*(\alpha) = 1 - \frac{(\alpha)_{n+1}}{(n+1)!},$$

$$(3.23) \quad Q_n^*(\alpha) + \frac{1}{2} Q_{n-1}^*(\alpha) + \cdots + \frac{1}{n+1} Q_0^*(\alpha) = P_{n+1}^*(\alpha) + \frac{(\alpha-1)_{n+2}}{(n+1)!},$$

$n=0, 1, 2, \dots$

Proof. Let us consider the function

$$(3.24) \quad F(t, z) = (1-z)^{-t} = \sum_{n=0}^{\infty} \frac{(t)_n}{n!} z^n,$$

where $(t, z) \in D_{iz} \equiv \{(t, z) | t \in K; -1 < z < 1\}$.

It is clear that the series (3.24) is uniformly convergent with respect to $t \in K$. It follows that the functions

$$(3.25) \quad \varphi(\alpha, z) = \int_{\alpha-1}^{\alpha} F(t, z) dt = -\frac{z(1-z)^{-\alpha}}{\ln(1-z)},$$

$$(3.26) \quad \psi(\alpha, z) = \int_{\alpha-1}^{\alpha} (\alpha-t) F(t, z) dt = -\frac{\varphi(\alpha, z) - (1-z)^{1-\alpha}}{\ln(1-z)},$$

$$(3.27) \quad \varphi^*(\alpha, z) = \int_{\alpha-1}^0 F(t+1, z) dt = -\frac{(1-z)^{-1} - (1-z)^{-\alpha}}{\ln(1-z)},$$

$$(3.28) \quad \psi^*(\alpha, z) = \int_0^{\alpha-1} t F(t+1, z) dt = -\frac{\varphi^*(\alpha, z) + (1-\alpha)(1-z)^{-\alpha}}{\ln(1-z)},$$

are continuous, having continuous derivatives in D_{iz} . According to the formulae (3.25)–(3.28) and (3.7), (3.11) we can write

$$-\varphi(\alpha, z) \frac{\ln(1-z)}{z} = (1-z)^{-\alpha}; \quad -\psi(\alpha, z) \ln(1-z) = \varphi(\alpha, z) - (1-z)^{1-\alpha};$$

$$-\varphi^*(\alpha, z) \ln(1-z) = (1-z)^{-1} - (1-z)^{-\alpha};$$

$$-\psi^*(\alpha, z) \ln(1-z) = (\alpha-1)(1-z)^{-\alpha} + \varphi^*(\alpha, z),$$

from which it follows

$$(3.29) \quad \left(\sum_{k=0}^{\infty} \frac{z^k}{k+1} \right) \left(\sum_{n=0}^{\infty} P_n(\alpha) z^n \right) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} z^n,$$

$$(3.30) \quad \left(\sum_{k=1}^{\infty} \frac{z^k}{k} \right) \left(\sum_{n=0}^{\infty} Q_n(\alpha) z^n \right) = \sum_{n=1}^{\infty} \left(P_n(\alpha) - \frac{(\alpha-1)_n}{n!} \right) z^n,$$

$$(3.31) \quad \left(\sum_{k=1}^{\infty} \frac{z^k}{k} \right) \left(\sum_{n=0}^{\infty} P_n^*(\alpha) z^n \right) = \sum_{n=1}^{\infty} \left(1 - \frac{(\alpha)_n}{n!} \right) z^n,$$

$$(3.32) \quad \left(\sum_{k=1}^{\infty} \frac{z^k}{k} \right) \left(\sum_{n=0}^{\infty} Q_n^*(\alpha) z^n \right) = \sum_{n=1}^{\infty} \left(\frac{(\alpha-1)_{n+1}}{n!} + P_n^*(\alpha) \right) z^n.$$

From the identities (3.29)–(3.32) it follows the recurrence relations (3.20)–(3.23).

4. A Numerical Example. As an illustration of the above, we shall apply an (αAA^*) -algorithm for the numerical solution of the following initial value problem

$$(4.1) \quad \ddot{x} = \frac{12\sqrt{1-t^2} + 4t\dot{x} - x}{4(1-t^2)}, \quad x(0)=0, \quad \dot{x}(0)=2, \quad -1 < t < 1,$$

with the accurate solution

$$(4.2) \quad x(t) = \frac{4t(1+t)}{\sqrt{1+t} + \sqrt{1-t^2}}.$$

To this end, we shall use the $(0.5AA^*)$ -algorithm for $p=2$ defined by the formulae

$$(\bar{A}) \quad \begin{cases} \dot{x}_k = \dot{x}_{k-1} + \frac{h}{24} (25\ddot{x}_{k-0.5} - 2\ddot{x}_{k-1.5} + \ddot{x}_{k-2.5}), \\ x_k = x_{k-1} + h\dot{x}_{k-1} + \frac{h^2}{48} (19\ddot{x}_{k-0.5} + 6\ddot{x}_{k-1.5} - \ddot{x}_{k-2.5}), \end{cases}$$

$$(\bar{A}^*) \quad \begin{cases} \dot{x}_{k-0.5} = \dot{x}_{k-1} + \frac{h}{24} (29\ddot{x}_{k-1.5} - 25\ddot{x}_{k-2.5} + 8\ddot{x}_{k-3.5}), \\ x_{k-0.5} = x_{k-1} + \frac{h}{2}\dot{x}_{k-1} + \frac{h^2}{384} (107\ddot{x}_{k-1.5} - 86\ddot{x}_{k-2.5} + 27\ddot{x}_{k-3.5}), \end{cases}$$

where

$$(4.3) \quad \ddot{x}_{j-0.5} = \frac{12\sqrt{1-t_{j-0.5}^2} + 4t_{j-0.5}\dot{x}_{j-0.5} - x_{j-0.5}}{4(1-t_{j-0.5}^2)}, \quad j = -2, -1, 0, \dots$$

The starting set for the (0.5AA)-algorithm is

$$(4.4) \quad \mathcal{M}_{AA^*}^{(1)}(0.5) = \{\tilde{x}_0 = 0, \tilde{x}_0 = 2, \tilde{x}_{-0.5} = -0.0198, \tilde{x}_{-0.5} = 1.970, \\ \tilde{x}_{-1.5} = -0.059, \tilde{x}_{-1.5} = 1.911, \tilde{x}_{-2.5} = -0.096, \tilde{x}_{-2.5} = 1.852\}.$$

Results computed with $h=0.02$ and the committed errors are given in Table 1.

k	$t_{k-0.5}$	$x_{k-0.5}$	$x_{k-0.5}$	$x_{k-0.5}$	t_k	x_k	x_k	$x(t_k)$	$e_k = x_k - x(t_k) $
-2	-0.05	1.852	-0.096	2.9351					
-1	-0.03	1.911	-0.059	2.9587					
0	-0.01	1.970	-0.0198	2.9854	0	2	0	0	0
1	0.01	2.03008	0.02015	3.01541	0.02	2.06031	0.04065	0.04060	$5 \cdot 10^{-5}$
2	0.03	2.09071	0.06140	3.04876	0.04	2.12129	0.08246	0.08241	$5 \cdot 10^{-5}$
3	0.05	2.15205	0.10383	3.08561	0.05	2.18300	0.12545	0.12546	10^{-5}
4	0.07	2.21416	0.14744	3.12609	0.08	2.24552	0.16973	0.16974	10^{-5}
5	0.09	2.27711	0.19234	3.17036	0.10	2.30893	0.21527	0.21528	10^{-5}
6	0.11	2.34099	0.23852	3.21862	0.12	2.37330	0.26209	0.26210	10^{-5}

In another Note we shall analyse the convergence and the numerical stability of the methods of the family (αAA^*).

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O FAMILIE DE METODE CU MAI MULȚI PAȘI PENTRU ECUAȚII DIFERENȚIALE DE ORDINUL DOI

(Rezumat)

Se stabilește o familie de metode cu mai mulți pași (αAA^*), pentru rezolvarea numerică a problemelor de tipul (1.1), (1.2).

ON THE MOMENTS OF ORTHOGONAL POLYNOMIALS

BY

ALLAN M. KRALL

1. Introduction. Given a set of moments $\{\mu_n\}_{n=0}^{\infty}$, where

$$\Delta_n = \begin{vmatrix} \mu_0 & \dots & \mu_n \\ & \ddots & \\ \mu_n & \dots & \mu_{2n} \end{vmatrix} \neq 0, \quad n=0, 1, \dots,$$

a set of polynomials $p_0=1$,

$$p_n = \begin{vmatrix} \mu_0 & \dots & \mu_n \\ \mu_{n-1} & \dots & \mu_{2n-1} \\ 1 & \dots & x^n \end{vmatrix} \div \Delta_{n-1}, \quad n=1, \dots,$$

is said to be orthogonal. For such polynomials a distributional weight function w exists [2],

$$w = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mu_n \delta^{(n)}(x),$$

with the property that $\langle w, p_0^2 \rangle = \mu_0$, $\langle w, p_n^2 \rangle = \Delta_n / \Delta_{n-1}$, $n=1, \dots$, and $\langle w, p_n p_m \rangle = 0$, $n \neq m$, in the sense of distributions.

Given such a set $\{\mu_n\}_{n=0}^{\infty}$ it is easy to calculate the coefficients a_{nj} of x^j in p_n in terms of the moments $\{\mu_n\}_{n=0}^{\infty}$.

Conversely, however, given a set of polynomials

$$p_n(x) = \sum_{j=0}^n a_{nj} x^j, \quad n=0, 1, \dots,$$

it is not nearly so easy to determine if the set is orthogonal or to calculate the moments in terms of the coefficients $\{a_{nj}\}_{n=0}^{\infty}$. In this paper we propose to give a solution to this converse problem, and to establish some consis-

tency formulas which must hold if the polynomials $\{p_n\}_{n=0}^{\infty}$ are to be orthogonal.

2. The Moments. For simplicity we assume that $\mu_0=1$. We also assume that the polynomials

$$p_n(x) = \sum_{j=0}^n a_{nj} x^j, \quad n=0, 1, \dots,$$

are orthogonal so that the weight function

$$w = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mu_n \delta^n(x)$$

exists. Since p_n is orthogonal to p_0 when $n \geq 1$, we find $\langle w, p_n \rangle = 0$, $n=1, \dots$, or

$$\sum_{j=1}^n a_{nj} \mu_j = -a_{n0}, \quad n=1, \dots$$

spelled out they are

$$a_{11} \mu_1 = -a_{10},$$

$$a_{21} \mu_1 + a_{22} \mu_2 = -a_{20},$$

$$a_{31} \mu_1 + a_{32} \mu_2 + a_{33} \mu_3 = -a_{30},$$

.....

These are easily solved to yield

Theorem. If the polynomials $\{p_n\}_{n=0}^{\infty}$ are orthogonal, then

$$\mu_n = \frac{(-1)^n \det(a_{i,j-1})_n}{\prod_{i=1}^n a_{ii}},$$

where $a_{ij}=0$, $j > i$ and

$$\det(a_{i,j-1})_n = \begin{vmatrix} a_{10} & \dots & a_{1,n-1} \\ a_{20} & \dots & a_{2,n-1} \\ \vdots & \ddots & \vdots \\ a_{n0} & \dots & a_{n,n-1} \end{vmatrix}.$$

3. Consistency Conditions. The preceding calculation can be performed for any set of polynomials for which p_n is exactly of degree n . Unfortunately not all are orthogonal. Consequently, the coefficients $\{a_{nj}\}_{n,j=0}^{\infty}$ must satisfy additional constraints.

Since p_n is supposed to be orthogonal to polynomials of degree less than n , and hence to x^m , $m < n$, we multiply p_n by x^m and apply w . This results in

$$0 = \sum_{j=0}^n a_{nj} \mu_{m+j}, \quad m=0, \dots, n-1.$$

If $m+n=p$, we find

Theorem. A necessary and sufficient condition for a set of polynomials

$$p_n(x) = \sum_{j=0}^n a_{nj} x^j, \quad n=0, 1, \dots,$$

to be orthogonal is that

$$0 = \sum_{j=0}^n a_{nj} \mu_{p-n+j},$$

where $(p+1)/2 \leq n \leq p$, $p=1, 2, \dots$

The equation with $n=p$ is that used to calculate μ_n . Those for which $n < p$ are necessary in order to have orthogonality. Note that the number of equations (n) increases far more rapidly than the number of moments (p).

Corollary. A necessary and sufficient condition for a set of polynomials

$$p_n(x) = \sum_{j=0}^n a_{nj} x^j, \quad n=0, 1, \dots,$$

to be orthogonal is that

$$0 = \frac{\sum_{k=0}^n a_{nk} (-1)^{p-n+k} \det(a_{i,j-1})_{p-n+k}}{\prod_{i=1}^{p-n+k} a_{ii}},$$

$$\frac{1}{2}(p+1) \leq n \leq p, \quad p=1, 2, \dots$$

4. An Example: the Brafman Polynomials [1]. In 1957 F. Brafman defined the polynomials

$$g_n(x) = \sum_{k=0}^{\frac{n}{p}} \frac{(-1)^k (n!)^q \prod_{i=1}^r (a_i)_k x^{n-pk}}{((n-pk)!)^q \prod_{i=1}^s (b_i)_k k!}, \quad n=0, 1, \dots,$$

where $(a)_n = a(a+1) \dots (a+n-1)$, while defining some generating functions for the Laguerre and Hermite polynomials. For purposes of simplicity let us assume that $p=q=r=s=1$. Then $\mu_0=1$, $\mu_1=a/b$, $\mu_2=(a^2b+2a^2-ab)/b^2(b+1)$ and

$$\mu_3 = \frac{a^3b^2+2ab^2-3a^2b^2-12a^2b+6a^3b+12a^3}{b^3(b+1)(b+2)}.$$

The first consistency condition, however,

$$\sum_{j=0}^n a_{nj} \mu_{p-n+j} = 0,$$

with $p=3$, $n=2$, yields

$$\frac{a(a+1)}{b(b+1)} \mu_1 - \frac{2a}{b} \mu_2 + \mu_3 = 0.$$

A quick check reveals this to be false, and so the Brafman polynomials are not orthogonal.

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ASUPRA MOMENTELOR POLINOAMELOR ORTOGONALE

(Rezumat)

Dat fiind un șir de polinoame ortogonale $\{p_n\}_{n=0}^{\infty}$, este ușor de calculat coeficienții a_{nj} ai puterilor x^j din p_n în funcție de momentele $\{\mu_n\}_{n=0}^{\infty}$ ale acestor polinoame. Autorul expune soluția problemei inverse stabilind dacă un șir dat de polinoame este sau nu ortogonal și, în cazul afirmativ, calculează momentele corespunzătoare în funcție de coeficienții $\{a_{nj}\}_{m,n=0}^{\infty}$ acestor polinoame ortogonale. Criteriile și formulele stabilite se aplică pentru a demonstra că polinoamele lui Brafman [1] nu sînt ortogonale.

ON THE DIRECTRIX CURVES AND PANGEODESICS OF A RULED SURFACE

BY

FROIM MARCUS

1. Introduction. Some new properties of the directrix curves of ruled surfaces are proved. It is shown that only ruled surfaces belonging to linear complexes admit conjugate nets all of whose curves are Fubini pangeodesics. $\mathcal{D}$ -surfaces are isolated, and other ruled surfaces are obtained in finite form.

Let S be a ruled surface generated by the point $x=x(u, v)$ where u, v are asymptotic parameters. It may be supposed (see [1], [2]), that

$$(1.1) \quad x = y + uz,$$

$y=y(v), z=z(v)$ being the directrix curves of S . From (1.1) results

$$(1.2) \quad x_{uv} = 0,$$

and from the general theory of surfaces it is known that the x -coordinates of a surface are solutions of a complete integrable system

$$(1.3) \quad x_{uu} = \theta_u x_u + \beta x_v + p_{11} x; \quad x_{vv} = \gamma x_u + \theta_v x_v + p_{22} x.$$

Since $x_{uv} = 0$, we have

$$(1.4) \quad \beta = \theta_u = p_{11} = 0,$$

because x, x_u, x_v are independent. It thus follows from (1.3) that

$$(1.3') \quad x_{uu} = 0; \quad x_{vv} = \gamma x_u + \theta_v x_v + p_{22} x,$$

and taking into account (1.1)

$$(1.5) \quad y'' + uz'' = \gamma z + \theta'(y' + uz') + p_{22}(y + uz).$$

Differentiating twice with respect to u , we have

$$(1.6) \quad (\gamma_{uu} + 2p_{22u} + up_{22uu})z + p_{22uu}y = 0,$$

whence

$$(1.7) \quad \gamma_{uu} + 2p_{22u} = 0; \quad p_{22uu} = 0,$$

which are the integrability conditions of (1.3').

Integration of (1.7) yields

$$(1.8) \quad \gamma = A + 2Bu + Cu^2; \quad p = p_{22} = -(j + B + Cu),$$

A, B, C, j being functions of v alone. The x_i , or (which is the same) the y_i and z_i ($i=1, 2, 3, 4$), may be multiplied by a factor so as to yield $\theta = \text{constant}$. Hence

$$(1.9) \quad x_{vv} = \gamma z + p x.$$

Differentiating with respect to u , and taking into account (1.1), we have

$$(1.10) \quad z' = (\gamma_u + p) z + p_u x.$$

But by (1.9')

$$(1.9') \quad z = \frac{x_{vv} - p x}{\gamma}, \quad (\gamma \neq 0),$$

whence

$$(8) \quad \frac{\partial^2}{\partial v^2} \left(\frac{x_{vv} - p x}{\gamma} \right) - (p + \gamma_u) \left(\frac{x_{vv} - p x}{\gamma} \right) - p_u x = 0,$$

is the differential equation of the asymptotes $u = \text{const.}$

2. Some Properties of a Class of Ruled Surfaces. An important class of ruled surfaces are those, every asymptotic curve of which belongs to a linear complex. In [1, p. 113] the following theorem is proved: *If every non rectilinear asymptotic curve of a ruled surface belongs to a linear complex, all these curves are projectively equivalent.*

The converse of the above was formulated (and proved) by G. Fubini in [3], as follows: *If all non rectilinear asymptotic curves of a ruled surface are mutually projective, every one of them belongs to a linear complex.*

All these surfaces must satisfy the condition:

$$(2.1) \quad \frac{\partial \log \gamma}{\partial v} = F(v), \quad \left(\frac{\partial \log \beta}{\partial u} = \psi(u) \right).$$

A ruled surface S belongs to a linear complex, if

$$(2.2) \quad k = \begin{vmatrix} A & B & C \\ A' & B' & C' \\ A'' & B'' & C'' \end{vmatrix} = 0,$$

(see [1], §35B), and

$$(2.3) \quad \frac{A'}{A} = \frac{B'}{B} = \frac{C'}{C},$$

are the conditions for S to belong to a linear congruence ([1], §37, p. 223). A ruled surface belongs to a linear complex also if

$$(2.4) \quad \frac{\partial^2 \log \gamma}{\partial u \partial v} = 0, \quad \left(\frac{\partial^2 \log \beta}{\partial v^2} = 0 \right).$$

Taking into account (1.8), we have exactly the conditions (2.3). Integration of the latter yields the general expression:

$$(2.5) \quad \gamma = (k_1 + k_2 u + k_3 u^2) V,$$

where $V = V(v)$, k_i are constants, and whence

$$\frac{\partial \log \gamma}{\partial v} = F(v), \quad \left(\frac{\partial \log \beta}{\partial u} = \psi(u) \right).$$

3. A New Property of the Directrix Curves of a Ruled Surface. None of the sources [1], [2] or [4] presents an interpretation of the elements of (2.3), namely:

$$(3_1) \quad \frac{A'}{A} = \frac{B'}{B} \quad \text{and} \quad (3_2) \quad \frac{C'}{C} = \frac{B'}{B},$$

which together formed the condition (2.3).

To remedy this shortage, we will prove the following

Theorem. *If either of the conditions (σ_1) or (σ_2) is satisfied, then one of the directrix curves of the ruled surface belongs to a linear complex.*

To prove the theorem, we observe that by (1.5), (1.8) and (1.9),

$$(3.1) \quad y'' - Az + (j+B)y = 0; \quad z'' + (j-B)z + Cy = 0.$$

Eliminating z , we obtain

$$(\delta') \quad \left(\frac{y'' + (j+B)y}{A} \right)' + (j-B) \left(\frac{y'' + (j+B)y}{A} \right) + Cy = 0, \quad (A \neq 0)$$

which is the differential equation of the directrix curve $y=y(v)$.

Using Wilczynski's notation [4]¹, (δ') may be rewritten as

$$(\delta'') \quad y'''' + 4p_1 y''' + 6p_2 y'' + 4p_3 y' + p_4 y = 0,$$

where

$$(3.2) \quad p_1 = -\frac{A'}{2A}; \quad p_2 = \frac{1}{3} \left(\frac{A'}{A} \right)^2 - \frac{1}{6} \frac{A''}{A} + \frac{1}{3} j; \quad p_3 = \frac{j' + B'}{2} - \left(\frac{j+B}{2} \right) \frac{A'}{A};$$

$$p_4 = j'' + B'' - 2(j' + B') \frac{A'}{A} - (j+B) \frac{A''}{A} + 2(j+B) \left(\frac{A'}{A} \right)^2 + j^2 - B^2 + AC,$$

and the invariants are

$$(3.3) \quad P_2 = p_2 - p_1^2; \quad P_3 = p_3 - p_1'' - 3p_1 p_2 + 2p_1^3;$$

$$P_4 = p_4 - 4p_1 p_3 - 3p_2^2 + 12p_1^2 p_2 - 6p_1^4 - p_1^{(3)};$$

and⁽²⁾

$$(\rho) \quad \Omega_3^{(2)} = P_3 - \frac{3}{2} P_2'; \quad \Omega_4 = P_4 - 2P_3' + \frac{6}{5} P_2'' - \frac{6}{25} P_2^2.$$

Putting $y = A^{1/2} Y$, following Fubini [3], equation (δ'') becomes

$$(\delta''') \quad Y'''' + lY'' + mY' + nY = 0.$$

One of the projective invariants of (δ''') is given by [3]

$$(\bar{\rho}) \quad (l' - m) dv^3.$$

To prove our theorem, it suffices to write

$$(\bar{\rho}') \quad l' - m' = 0.$$

But for the pattern below we use (ρ) .

Taking into account (3.2), we have

$$(3.3') \quad P_2 = \frac{1}{3} j - \frac{5}{12} \left(\frac{A'}{A} \right)^2 + \frac{1}{3} \frac{A''}{A};$$

$$P_3 = \frac{j'}{2} + \frac{B'}{2} - \frac{BA'}{2A} + \frac{A'''}{2A} - \frac{7}{4} \frac{A'A''}{A^2} + \frac{5}{4} \left(\frac{A'}{A} \right)^2.$$

¹ This notation was used first by Halphen (see [9]).

² The invariant Ω_3 of a linear differential equation of the third order was found first by Halphen [9].

Therefore

$$(\rho') \quad \Omega_3 = \frac{AB' - BA'}{2A}.$$

Again according to Wilczyński (ibid., p. 243) a space curve belongs to a linear complex iff

$$(\rho'') \quad \Omega_3 = 0.$$

Therefore, iff

$$(\sigma_1) \quad \frac{A'}{A} = \frac{B'}{B},$$

the directrix curve $y=y(v)$ belongs to a linear complex, and the converse also holds. Eliminating y from (3.1), we have:

$$(3.4) \quad \left(\frac{z'' + (j-B)z}{C} \right)'' + \left(\frac{j+B}{C} \right) z'' + \frac{j^2 - B^2}{C} z + Az = 0.$$

The above is obtained directly from (δ') if A is replaced with C , and B with $-B$. Therefore, the directrix curve $z=z(v)$ belong to a linear complex, iff

$$(\sigma_2) \quad \frac{C'}{C} = \frac{B'}{B}. \quad \text{Q.E.D.}$$

Consequently: *If both directrix curves of a ruled surface belong to a linear complex, then S belongs to a linear congruence.*

Integration of (σ_1) yields

$$(3.5) \quad A = kV; \quad B = k_1V, \quad (k, k_1 = \text{const.})$$

whence $\gamma = (k + 2k_1u)V + Cu^2$; $p = -j - k_1V - Cu$, and similarly from (σ_2)

$$(3.6) \quad \gamma = A + (2k_2u + k_3u^2)V; \quad p = -j - (k_2 + k_3u)V, \quad (k_2, k_3 = \text{const.}).$$

It should be noted here that according to Fubini [5], a non-ruled surface can admit at most a group G_2 of two parameters of projective deformation into itself. Only ruled surfaces such that $\gamma(\beta)$ can be reduced to a constant, admit a group G_3 of p.d. into themselves; these are exactly the surfaces belonging to a linear complex or to a linear congruence.

4. The Pangeodesics of a Ruled Surface. E. Cartan proved in [6] the existence of surfaces for which there exist conjugate nets such that

$$\bar{h} = h; \quad \bar{k} = k$$

($h, k, \bar{h}, \bar{k}$ being respectively the Darboux point — and tangential invariants of the corresponding Laplace equation), and which he called $\mathcal{L}$ -surfaces and $\mathcal{L}$ -nets. We refer to the former as Cartan-Terracini surfaces, the strength of another of their important properties — observed by A. Terracini — regarding projective applicability of congruences generated by the tangents of an $\mathcal{L}$ -net. In [7], we characterized the $\mathcal{L}$ -surfaces by the following properties:

(a) If the curves of a conjugate net (Γ) of a non-ruled surface are pangeodesics, S is an $\mathcal{L}$ -surface and Γ an $\mathcal{L}$ -net.

(b) Only for $\mathcal{L}$ -surfaces there exist conjugate nets whose curves are Fubini pangeodesics.

Let S be a ruled surface determined by

$$(4.1) \quad x_{uu}=0; \quad x_v=\gamma x_u+p x,$$

with γ and p as per (1.8). Observing that the Fubini linear element of S is

$$(4.2) \quad \frac{\beta du^3 + \gamma dv^3}{2du dv} = \gamma \frac{dv^2}{2du},$$

then the differential equation of the pangeodesics, (see [1, 2] reads

$$(4.3) \quad v'' + (\ln \gamma)_u v' + \frac{1}{2} (\ln \gamma)_v v'^2 = 0; \quad v' = \frac{dv}{du}.$$

Suppose that the curves of (Γ) determined by

$$(4.4) \quad dv^2 - \tau^2 du^2 = 0; \quad (\tau = \tau(u, v))$$

are pangeodesics, they must then satisfy (4.3), and in such a case it follows that $\partial^2 \ln \gamma / \partial u \partial v = 0$. Hence S belongs to a linear complex, and (Γ) is an isotherm-conjugate net.

By (1.8) we have

$$(4.5) \quad \frac{A'}{A} = \frac{B'}{B} = \frac{C'}{C},$$

hence

$$(4.6) \quad \gamma = (k + 2k_1 u + k_2 u^2) V; \quad p = -j - k_1 - k_2 u,$$

where $V = V(v)_1$ and k_i are constants. Conversely, from (4.6) it follows that if

$$(4.7) \quad \tau = \pm \frac{dv}{du} = \frac{\pm C}{(k + 2k_1 u + k_2 u^2) \sqrt{V}},$$

the curves of the net $dv^2 - \tau^2 du^2 = 0$ satisfy (4.3). Accordingly, the only ruled surface which belongs to a linear complex admit conjugate nets all of whose curves are pangeodesics.

5. The Ruled Surfaces for $\gamma = A$, $\gamma = 2Bu$, $\gamma = Cu^2$ (A, B, C constants). On the basis of an observation by Fubini [5], we determined in [8], in finite form, the ruled surfaces corresponding to $\gamma = A = k$, $k = \text{const.}$ In this case

$$(5.1) \quad p = -j.$$

From (δ) and (δ') it follows that

$$(5.2) \quad x'''' + 2jx'' + 2j'x' + (j'' + j^2)x = 0,$$

$$(5.3) \quad y'''' + 2jy'' + 2j'y' + (j'' + j^2)y = 0,$$

and

$$(5.4) \quad z'' + jz = 0.$$

The directrix $z=z(v)$ is a straight line. It is seen that when γ is reduced to a constant, the differential equation of the asymptotic curves $u=\text{const.}$ coincides with that of the directrix curve $y=y(v)$, and depends on $j=j(v)$ alone.

In [1, p. 219], it is shown that if

$$(5.5) \quad j=0,$$

then the corresponding ruled surfaces admit ∞^3 collineations into themselves; this, according to Fubini's results [5]. For proof, one suppose $\gamma=A=1$, $B=C=0$, $\theta'=0$, $j=0$, hence $y''=z$, $z''=0$. Integration yields:

$$z(0, 0, 1, v); \quad y\left(1, v, \frac{v^2}{2}, \frac{v^3}{6}\right); \quad x\left(1, v, \frac{v^2}{2}+u; \frac{v^3}{6}+uv\right),$$

and in non-homogeneous coordinates $z=xy-x^3/3$.

From (5.3), (8''), (3.2) and (3.3), noting that $\gamma=A=k$ we have

$$(5.6) \quad p_1=0; \quad p_2=\frac{1}{3}j; \quad p_3=\frac{1}{2}j'; \quad p_4=j''+j^2,$$

$$P_2=\frac{1}{3}j; \quad P_3=\frac{1}{2}j'; \quad P_4=j''+\frac{2}{3}j^2.$$

For surfaces S belonging to a linear complex, $\Omega_3=0$.

According to Wilczynski, a space curve is a twisted cubic iff

$$(\rho''') \quad \Omega_3=\Omega_4=0.$$

But

$$(5.7) \quad \Omega_4=P_4-2P_3'+\frac{6}{5}P_2''-\frac{6}{25}P_2^2=\frac{2}{5}\left(j''+\frac{8}{5}j^2\right).$$

Therefore the directrix curve $y=y(v)$ of the ruled surfaces determined by $\gamma=A=k$, are a twisted cubic iff

$$(5.8) \quad j''+\frac{8}{5}j^2=0, \quad (j=j(v)).$$

As the asymptotes $u=\text{const.}$ are in this case projectively equivalent to the directrix $y=y(x)$, we have the following result: *The not-straight asymptotes of the ruled surfaces corresponding to $\gamma=\text{const.}$, are twisted cubics iff*

$$(\mathcal{F}) \quad j''+\frac{8}{5}j^2=0.$$

This condition is identically satisfied if $j=0$, and the corresponding surface is a Cayley cubic surface.

From $(\mathcal{F})$ it follows by integration that

$$j=\mathcal{Q}\left(\frac{2i}{\sqrt{15}}v+C_3\right)$$

where $\mathcal{Q}$ is a Weierstrass function, with $g_2=0$; $g_3=C_2$. We can suppose $p=j-ku$; therefore

$$\Omega_4 = \frac{2}{5} \left(-j'' + \frac{8}{5} j^2 \right),$$

and the not-straight asymptotes are twisted cubics if

$$j = \mathcal{Q} \left(\frac{2v}{\sqrt{15}} + C_3 \right); \quad g_2=0; \quad g_3=C_2.$$

We call ruled $\mathcal{D}$ -surfaces the surfaces determined by $\gamma=A=k$. They fall under four classes, $\mathcal{D}_{1b}$, $\mathcal{D}_2$, $\mathcal{D}_{2b}$ and $\mathcal{D}_{3b}$, and we obtained them in finite form (see [8]).

The $\mathcal{D}_1$ surfaces, which correspond to $j=0$, are generated by the point with coordinates

$$(5.9) \quad x^1=1; \quad x^2=v; \quad x^3=\frac{kv^2}{2}+u; \quad x^4=\frac{kv^3}{6}+uv$$

and in non-homogeneous coordinates by

$$(5.10) \quad z = xy - \frac{kx^3}{3},$$

and are Cayley cubic surfaces.

For $j=\alpha^2$ we have the $\mathcal{D}_{2b}$ surfaces. From

$$(5.11) \quad \begin{aligned} x_{uu} &= 0; \quad x_{vv} = kx_u - \alpha^2 x, \quad p = -j = -\alpha^2; \\ y'' - \alpha^2 y &= k; \quad z'' - \alpha^2 z = 0, \end{aligned}$$

we obtain

$$(5.12) \quad \begin{aligned} x^1 &= \sin \alpha v; \quad x^2 = \cos \alpha v; \quad x^3 = -\frac{kv}{2} + u \sin \alpha v; \\ x^4 &= \frac{kv}{2\alpha} \sin \alpha v + u \cos \alpha v, \end{aligned}$$

whence

$$(5.13) \quad z = xy - \frac{k}{2\alpha^2} (1+x^2) \tan^{-1} x.$$

If $j=-\alpha^2$ (α real), then

$$(5.14) \quad x^1 = e^{\alpha v}; \quad x^2 = e^{-\alpha v}; \quad x^3 = e^{\alpha v} \left(\frac{kv}{2\alpha} + u \right); \quad x^4 = e^{-\alpha v} \left(u - \frac{kv}{2\alpha} \right);$$

and

$$(5.15) \quad z = \frac{y}{x} - \frac{k}{2\alpha^2} \ln x,$$

yields the $\mathcal{D}_{2b}$ surfaces. Finally, let

$$(5.16) \quad p = j(v).$$

Hence

$$(5.17) \quad y'' + jy = kz; \quad z'' + jz = 0.$$

Let $F_1(v)$ and $F_2(v)$ be two linearly independent solutions of $z'' + jz = 0$. Denoting their Wronskian by $\Delta = \text{const.}$, we find

$$(5.18) \quad \begin{aligned} x^1 &= F_1(v); \quad x^2 = F_2(v); \quad x^3 = \frac{k}{\Delta} \left(F_2 \int F_1^2 dv - F_1 \int F_1 F_2 dv \right) + u F_1(v); \\ x^4 &= \frac{k}{\Delta} \left(F_2 \int F_1 F_2 dv - F_1 \int F_2^2 dv \right) + u F_2(v). \end{aligned}$$

Therefore

$$(5.19) \quad \Delta(x^2 x^3 - x^1 x^4) = k \left((x^1)^2 \int (x^2)^2 dv - (x^2)^2 \int (x^1)^2 dv \right).$$

For further results, see [8].

6. The Case $A' = B' = C' = 0$. In this case

$$(6.1) \quad A = k_1; \quad B = k_2 = C = k_3; \quad k_i = \text{const.}$$

$$\gamma = k_1 + 2k_2 u + k_3 u^2; \quad p = -j - k_2 - k_3 u.$$

Hence

$$(6.2) \quad y'' + (j + k_2)y = k_1 z; \quad z'' + (j - k_2)z = -k_3 y.$$

From (8''), (3.2), (3.4) and (8), it follows that

$$(6.3) \quad \begin{aligned} y'''' + 2jy'' + 2j'y' + (j'' + j^2 + k_1 k_3 - k_2^2)y &= 0; \\ z'''' + 2jz'' + 2j'z' + (j'' + j^2 + k_1 k_3 - k_2^2)z &= 0; \\ x'''' + 2jx'' + 2j'x' + (j'' + j^2 + k_1 k_3 - k_2^2)x &= 0. \end{aligned}$$

Consider now the case

$$(6.4) \quad \gamma = ku^2; \quad p_1 = -j - ku.$$

This means $k_1 = k_2 = 0$, $k_3 = k$, whence

$$(6.2') \quad y'' + jy = 0; \quad z'' + jz = -ky.$$

Now the directrix y is a straight line.

Comparing (6.3) with (5.2), it is seen that the differential equation in x coincides with (6.3) in z . Therefore we obtain the same $\mathcal{D}$ -surfaces even though the coordinates differ. This indicates that change of the parameters and normalization may yield $\gamma = k = \text{const.}$

For $j = 0$,

$$(6.4') \quad x^1 = u; \quad x^2 = uv; \quad x^3 = 1 - \frac{k}{2}uv^2; \quad x^4 = v - \frac{k}{6}uv^3,$$

whence, denoting $x = v$, $y = \frac{1}{u} - \frac{kv^2}{2}$, $z = \frac{v}{u} - \frac{kv^3}{6}$, we obtain

$$(6.5) \quad z = xy + \frac{k}{3} x^3,$$

which is again Cayley ruled surface 1.

Let $j = -\alpha^2$. In this case

$$(6.6) \quad x^1 = ue^{\alpha v}; \quad x^2 = ue^{-\alpha v}; \quad x^3 = e^{\alpha v} \left(1 - \frac{kuv}{2\alpha}\right); \quad x^4 = e^{-\alpha v} \left(1 + \frac{kuv}{2\alpha}\right).$$

Putting

$$(6.7) \quad x = e^{-\alpha v}; \quad y = e^{-2\alpha v} \left(\frac{1}{u} + \frac{kv}{2\alpha}\right); \quad z = \frac{1}{u} - \frac{kv}{2\alpha};$$

we obtain

$$(6.8) \quad \frac{y}{x} = \frac{1}{u} + \frac{kv}{2\alpha}; \quad z = \frac{1}{u} - \frac{kv}{2\alpha}$$

whence

$$(6.9) \quad z = \frac{y}{x} + \frac{k}{2\alpha^2} \ln x.$$

Finally, for $j = \alpha^2$ we obtain

$$(6.10) \quad x^1 = u \sin \alpha v; \quad x^2 = u \cos \alpha v; \quad x^3 = \sin \alpha v + \frac{kuv}{2\alpha} \cos \alpha v;$$

$$x^4 = \cos \alpha v - \frac{kuv}{2\alpha} \sin \alpha v.$$

Denoting

$$(6.11) \quad x = \tan \alpha v; \quad y = \frac{1}{u} - \frac{k}{2\alpha^2} x \tan^{-1} x; \quad z = \frac{1}{u} x + \frac{k}{2\alpha} \tan^{-1} x,$$

we have

$$(6.12) \quad z = xv + \frac{k}{2\alpha^2} (1+x^2) \tan^{-1} x.$$

It should be noted that if $\gamma = ku^2$, $p = -j - ku$ ($k, j = \text{const.}$), the corresponding surfaces are obtained from the $\mathcal{D}$ -surfaces, equivalent to replacing k with $-k$.

Let now

$$(6.13) \quad A = C = 0; \quad B = k^2.$$

Then

$$(6.14) \quad \gamma = 2k^2 u; \quad p = -j - k^2,$$

and by (3.1)

$$(6.15) \quad y'' + (j + k^2)y = 0; \quad z'' + (j - k^2)z = 0.$$

The directrices are straight lines. The differential equation of the asymptotic curves $u = \text{const.}$ reads

$$(6.16) \quad x'''' + 2jx'' + (j^2 + j'' - k^4)x = 0$$

and from (3.3) and (ρ) it follows that

$$(6.17) \quad \Omega_1 = \frac{2}{5} \left(j'' + \frac{8}{5} j^2 \right) - k^4.$$

Therefore, in this case, for $j=0$, the asymptotic curves cannot be skew cubics.

For $j=0$, we have the coordinates

$$y(\cos kv, \sin kv, 0, 0); \quad z(0, 0, e^{kv}, e^{-kv})$$

and in non-homogeneous coordinates

$$(6.18) \quad x = \frac{e^{kv} \cos kv}{u}; \quad y = \frac{e^{kv} \sin kv}{u}; \quad z = e^{2kv},$$

whence

$$(6.19) \quad z = \exp \left(2 \tan^{-1} \frac{y}{x} \right).$$

If $B = -k^2$ (k real), the same surface is obtained.

We suppose now

$$(6.20) \quad j = -\alpha^2 \quad (\alpha = \text{const.}) \quad k^2 > \alpha^2.$$

In this case

$$(6.21) \quad y(\cos(\sqrt{k^2 - \alpha^2}v); \sin(\sqrt{k^2 - \alpha^2}v), 0, 0); \\ z(0, 0, \exp(\sqrt{k^2 + \alpha^2}v), \exp(-\sqrt{k^2 + \alpha^2}v))$$

and

$$(6.22) \quad x^1 = u \exp(\sqrt{k^2 + \alpha^2}v); \quad x^2 = u \exp(-\sqrt{k^2 + \alpha^2}v); \\ x^3 = \cos(\sqrt{k^2 - \alpha^2}v); \quad x^4 = \sin(\sqrt{k^2 - \alpha^2}v).$$

Hence

$$(6.23) \quad z = \exp \left(2 \sqrt{\frac{k^2 + \alpha^2}{k^2 - \alpha^2}} \tan^{-1} \frac{y}{x} \right).$$

If $k^2 < \alpha^2$, or $k^2 > \alpha^2$, we have the surfaces

$$(6.24) \quad z = \exp \left(2 \sqrt{\frac{k^2 - \alpha^2}{k^2 + \alpha^2}} \tan^{-1} \frac{y}{x} \right).$$

Supposing now that $j = \alpha^2$, $B = k^2$, then if $\alpha^2 = k^2$ we obtain

$$(6.25) \quad z = \frac{\sqrt{2}}{2\alpha} \tan^{-1} \frac{y}{x}.$$

Some observations. Let us consider again the Eqs. (3.1). By eliminating z we obtained the differential equation (8''). According to a *very important result of G. Koenigs* [9] we can find ruled surfaces, considering for example four algebraic independent functions y_i and to calculate the coefficient

ents p_1, p_2, p_3, p_4 . From (3.2) we find A, B, C, j ; then the coordinates z_4 . This results also, according to Koenigs [9], if $\Omega_3=0$; this is if a directrix belongs to a linear complex.

From (1.1) we obtain then the corresponding ruled surfaces.

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ASUPRA CURBELOR DIRECTOARE ȘI PANGEODEZICELOR UNEI SUPRAFEȚE RIGLATE

(Rezumat)

Se stabilesc noi proprietăți ale curbelor directoare ale suprafețelor riglate. Se arată că numai suprafețele riglate care aparțin complexelor liniare admit rețete conjugate avînd toate curbele pangeodezice Fubini și se dă forma finită a unui număr de suprafețe riglate remarcabile.

QUASI-CONNECTIONS ON DIFFERENTIABLE MANIFOLDS WITH Π -STRUCTURE

BY

ELENA VAMANU and IRINA CUIBARI

0. Introduction. Let M be a C^∞ -differentiable, n -dimensional manifold. A Π -structure (in the terminology of C. J. Hsu) is defined on M by giving a nondegenerate tensor field F_k^i such that

$$(1) \quad F_k^i F_j^k = \lambda^2 \delta_j^i$$

where λ is a nonnull complex constant and δ_j^i is the Kronecker symbol.

If M is a manifold admitting a Π -structure, we can introduce, following the way of [2], [3] the notion of quasi-connection on M .

Let Γ_{jk}^i be the quasi-connection coefficients, T_{ii}^k the torsion tensor and R_{hij}^k the curvature tensor of the quasi-connection. From [2], [3] we have:

$$(2) \quad T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k + \lambda^3 F_{h,m}^k (F_i^m F_j^h - F_j^m F_i^h),$$

$$(3) \quad R_{hij}^k = \Gamma_{im}^k \Gamma_{jh}^m - \Gamma_{jm}^k \Gamma_{ih}^m + \lambda \Gamma_{jh,m}^k F_i^m - \lambda \Gamma_{ih,m}^k F_j^m + \\ + \lambda^3 \Gamma_{ph}^k F_{a,b}^p (F_i^b F_j^a - F_j^b F_i^a).$$

The covariant differential of the tensor field F_j^i with respect to the quasi-connection Γ is given by

$$(4) \quad D_k F_j^i = \lambda F_{j,m}^i F_k^m + \Gamma_{km}^i F_j^m - \Gamma_{kj}^m F_m^i$$

(see [2], [3], [4], for details).

In the present paper we determine all the quasi-connections in which F_j^i is covariantly constant and we study some properties of these quasi-connections. The set of these quasi-connections is a group G .

In §1 we obtain a result on the integrability of a Π -structure. In §2 we find some invariants of the group G , and we give certain geometric interpretations for these invariants.

1. *F-II-quasi-connections*. Proceeding in the way as in [4] we can obtain all the *quasi-connections* in which F_j^i is covariantly constant. We shall call these *quasi-connections* *F-II-quasi-connections*. They are given by

$$(5) \quad \bar{\Gamma}_{jk}^i = F_j^m \left(\lambda \gamma_{mk}^i - \frac{1}{2\lambda} F_h^m \nabla_m F_h^i \right) + W_{jk}^i,$$

where γ_{mk}^i is an arbitrary but fixed connection on M , ∇ is the operator of covariant differentiation with respect to γ_{mk}^i , and W_{jk}^i is an arbitrary tensor field that satisfies the condition

$$(6) \quad Q_2 W_{jk}^i = 0,$$

where Q_2 denotes the Obata operator defined by

$$(7) \quad Q_2 W_{jk}^i = \frac{1}{2} \left(W_{jk}^i - \frac{1}{\lambda^2} F_a^i W_{jb}^a F_b^i \right).$$

In particular, if we denote

$$(8) \quad \Gamma_{jk}^i = F_j^m \left(\lambda \gamma_{mk}^i - \frac{1}{2\lambda} F_h^m \nabla_m F_h^i \right),$$

it can be easily verified that Γ_{jk}^i is an *F-II-quasi-connection*.

Let N_{jk}^i be the Nijenhuis tensor of the considered Π -structure, defined by

$$(9) \quad N_{jk}^i = \frac{1}{4\lambda^2} F_j^m (F_{m,k}^i - F_{k,m}^i) - F_k^m (F_{m,j}^i - F_{j,m}^i).$$

Let us now consider the quasi-connection Γ_{jk}^i given by (8) where γ_{mk}^i is an arbitrary symmetric connection on M . Let T_{jk}^i be the torsion tensor of the quasi-connection Γ_{jk}^i .

From (2), (8), and (9) we have

$$(10) \quad Q_3 T_{jk}^i = \lambda(2\lambda^4 - 1) F_j^m N_{km}^i,$$

where Q_3 denotes the Obata operator defined by

$$(11) \quad Q_3 T_{jk}^i = \frac{1}{2} \left(T_{jk}^i + \frac{1}{\lambda^2} F_j^a F_k^b T_{ab}^i \right).$$

If $2\lambda^4 - 1 \neq 0$ we have

Theorem 1. *The Π -structure given on M is integrable if and only if*

$$Q_3 T_{jk}^i = 0.$$

Proof. If the Π -structure is integrable we have $N_{km}^i = 0$, and from (11) we find $Q_3 T_{jk}^i = 0$. If $Q_3 T_{jk}^i = 0$, from (11) we have $F_j^m N_{km}^i = 0$, but F_j^m is a nondegenerate tensor, hence we obtain $N_{km}^i = 0$.

2. Invariants of the group G . Now if we consider the transformations $\Gamma_{jk}^i \rightarrow \bar{\Gamma}_{jk}^i$ defined by (5), W_{jk}^i being an arbitrary tensor field that satisfies (6), the set of these transformations is a group G , isomorphic to the additive group of the tensor field W . Next we take,

$$(12) \quad W_{jk}^i = \delta_k^i p_j.$$

One easily verifies that W defined by (12) satisfies (6) and hence we obtain

$$(13) \quad \bar{\Gamma}_{jk}^i = \Gamma_{jk}^i + \delta_k^i p_j.$$

The set of these transformations defines the subgroup G_1 of the group G .

Let T_{jk}^i , $\bar{T}_{jk}^i$ be the torsion tensors, and R_{hij}^k , $\bar{R}_{hij}^k$ the curvature tensors of the two F - Π -quasi-connections given by (13). Using (2), (3) and (13) we obtain

$$(14) \quad \bar{T}_{jk}^i = T_{jk}^i + \delta_k^i p_j - \delta_j^i p_k,$$

$$(15) \quad \bar{R}_{hij}^k = R_{hij}^k + \delta_h^k (D_i p_j - D_j p_i + T_{ij}^m p_m).$$

Proceeding in the same way as in [5] we find the next invariant tensors of the group G_1 :

$$(16) \quad {}_1I_{kh}^i = T_{kh}^i + \frac{1}{n-1} (\delta_k^i T_h - \delta_h^i T_k),$$

$$(17) \quad {}_2I_{hij}^k = R_{hij}^k - \frac{1}{n} \delta_h^k S_{ij},$$

$$(18) \quad {}_3I_{hij}^k = R_{hij}^k - \delta_h^k R_{ji},$$

where

$$(19) \quad T_k = T_{ki}^i, \quad S_{ij} = R_{kij}^k, \quad R_{hm} = R_{hmk}^k.$$

Now, we aim to give certain geometric interpretations for these invariants.

Theorem 2. *There always exists an F - Π -quasi-connection of the group G_1 with the torsion tensor ${}_1I_{kh}^i$.*

Proof. Let $\hat{\Gamma}_{jk}^i$ be the quasi-connection of the group G_1 given by

$$(20) \quad \hat{\Gamma}_{jk}^i = \Gamma_{jk}^i + \delta_k^i \hat{p}_j.$$

From (14) and (20) we have

$$(21) \quad \hat{T}_{jk}^i = T_{jk}^i + \delta_k^i \hat{p}_j - \delta_j^i \hat{p}_k.$$

On the other hand, from $\hat{T}_{jk}^i = {}_1I_{jk}^i$, we obtain

$$(22) \quad \delta_k^i \hat{p}_j - \delta_j^i \hat{p}_k = \frac{1}{n-1} (\delta_j^i T_k - \delta_k^i T_j).$$

If we contract i, k in (22), we have

$$n \hat{p}_j - \hat{p}_j = \frac{1}{n-1} (T_j - n T_j).$$

Hence we find

$$(23) \quad \hat{p}_j = - \frac{1}{n-1} T_j.$$

Substituting (22) in (20) we obtain the unique quasi-connection given by

$$(24) \quad \hat{\Gamma}_{jk}^i = \hat{\Gamma}_{jk}^i - \frac{1}{n-1} \delta_k^i T_j. \quad \text{Q. D. E.}$$

Remark. The curvature tensor of the quasi-connection $\hat{\Gamma}$ is of the form

$$(25) \quad \hat{R}_{hij}^k = R_{hij}^k - \frac{1}{n-1} \delta_h^k (D_i T_j - D_j T_i + T_{ij}^m T_m).$$

From (25) we have

Proposition. *The quasi-connections $\Gamma, \hat{\Gamma}$ have the same curvature tensor if and only if the torsion covector T_i satisfies the condition*

$$(26) \quad D_i T_j - D_j T_i + T_{ij}^m T_m = 0.$$

Let $\bar{\Gamma}_{jk}^i$ be a quasi-connection defined by (13). Now, we shall determine the conditions under which the quasi-connection considered has the curvature tensor given by ${}_2 I_{hij}^k$ or ${}_3 I_{hij}^k$. From (15) and (17) we have

$$\bar{R}_{hij}^k = R_{hij}^k + \delta_h^k (D_i p_j - D_j p_i + T_{ij}^m p_m) = {}_2 I_{hij}^k = R_{hij}^k - \frac{1}{n} \delta_h^k S_{ij},$$

from which we obtain

$$(27) \quad D_i p_j - D_j p_i + T_{ij}^m p_m = - \frac{1}{n} S_{ij}.$$

Thus we have

Theorem 3. *The tensor ${}_2 I_{hij}^k$ is the curvature tensor of the quasi-connection $\bar{\Gamma}_{jk}^i = \Gamma_{jk}^i + \delta_k^i p_j$ if and only if the covector p_j satisfies*

$$D_i p_j - D_j p_i + T_{ij}^m p_m = - \frac{1}{n} S_{ij}.$$

In the same way it can be obtained

Theorem 4. *The tensor ${}_3I_{hij}^k$ is the curvature tensor of the quasi-connection $\bar{\Gamma}_{jk}^i = \Gamma_{jk}^i + \delta_k^i p_j$ if and only if the covector p_j satisfies*

$$(28) \quad D_i p_j - D_j p_i + T_{ij}^m p_m = -R_{ji}.$$

Let $D, \bar{D}$ be the operators of covariant differentiation with respect to Γ and $\bar{\Gamma}$ respectively. We have then

$$(29) \quad \bar{D}_i p_j = \lambda p_{j,m} F_i^m - \bar{\Gamma}_{ij}^m p_m = \lambda p_{j,m} F_i^m - (i_{ij}^m + \delta_j^m p_i) p_m = D_i p_j - p_i p_j.$$

Using (29) we obtain an interesting property of the covector p_i , given by

$$(30) \quad \bar{D}_i p_j - \bar{D}_j p_i = D_i p_j - D_j p_i.$$

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CVASI-CONEXIUNI PE VARIETĂȚI DIFERENȚIABILE CU II-STRUCTURĂ

(Rezumat)

Dată o varietate diferențiabilă dotată cu II-structură, se determină toate cvasi-conexiunile în raport cu care tensorul II-structurii este covariant constant, numite *F-II-cvasi-conexiuni*. În partea a doua a lucrării se consideră o submulțime G_1 a mulțimii *F-II-cvasi-conexiunilor* care are structură de grup. Se determină o serie de invarianti ai acestui grup și se dau unele interpretări geometrice ale acestora.

ON THE GENERAL ALMOST SYMPLECTICAL FINSLER CONNECTIONS

BY

I. GHINĂ

0. The set of all general almost symplectical Finsler connections was determined in [1].

In the present paper we establish a new form able to express these connections and we determine the class of Finsler connections with the property that the associated Kawaguchi general almost symplectical connection coincides with the canonical general almost symplectical connection.

The definitions and notations which we are using are in general those from M. M a t s u m o t o's monograph [7].

1. Let M be a $2n$ -dimensional differentiable manifold. An almost symplectical Finsler structure on M is defined by a field of tensors $a_{ij}(x, y)$ (y being the support element in point x) with the properties

- (1) $a_{ij}(x, y) = -a_{ji}(x, y)$;
- (2) $\det(a_{ij}(x, y)) \neq 0, \quad \forall y \neq 0$.

Let associate to the almost symplectical structure $a_{ij}(x, y)$ the operators

$$(1.1) \quad \theta_{kh}^{ij} = \frac{1}{2} (\delta_k^i \delta_h^j - a_{kh} a^{ij}), \quad {}^* \theta_{kh}^{ij} = \frac{1}{2} (\delta_k^i \delta_h^j + a_{kh} a^{ij})$$

with their known properties [10].

We will now try to obtain the Finsler connections $FT = (N, F, C)$ with the properties

$$(1.2) \quad a_{ij|k} = H_{ijk}, \quad a_{ij|k} = K_{ijk},$$

where $H_{ijk}(x, y)$ and $K_{ijk}(x, y)$ are two given Finsler tensors of type $(0, 3)$, antisymmetric in the first two index. These connections are called in [3], general almost symplectical Finsler connections.

In order to determine this set of connections, we use the following method [9]: Let ${}^0 FT = ({}^0 N_k^i, {}^0 F_{jk}^i, {}^0 C_{jk}^i)$ be an arbitrary but fixed connection, defined on M . Then each connection $FT = (N_k^i, F_{jk}^i, C_{jk}^i)$ in M can be expressed as [7], [9]

$$(1.3) \quad \begin{cases} N_k^i = \overset{0}{N}_k^i - A_k^i, \\ F_{jk}^i = \overset{0}{F}_{jk}^i + \overset{0}{C}_{jm}^i N_k^m - B_{jk}^i, \\ C_{jk}^i = \overset{0}{C}_{jk}^i - D_{jk}^i, \end{cases}$$

where A_k^i , B_{jk}^i , D_{jk}^i denote the components of the tensor fields obtained as difference of the pair $(F\overset{0}{\Gamma}, F\overset{0}{\Gamma})$. We shall determine these fields A_k^i , B_{jk}^i , D_{jk}^i so that the connection given by (1.3) satisfies the relations (1.2). It results

$$(1.4) \quad \begin{cases} a_{hj} B_{ik}^h + a_{ih} B_{jk}^h = H_{ijk} - a_{ij|k}^\circ - A_k^h a_{ij|k}^\circ, \\ a_{hj} D_{ik}^h + a_{ih} D_{jk}^h = K_{ijk} - a_{ij|k}^\circ, \end{cases}$$

where $|$, $^\circ$ represents the covariant h -derivative, respectively v -derivative with respect to the fixed connection $F\overset{0}{\Gamma}$.

Let multiply by $\frac{1}{2}a^{ij}$ the relations (1.4) and put in evidence the operator $^*\theta_{hj}^i$, we get

$$(1.5) \quad \begin{cases} ^*\theta_{hj}^i B_{ik}^h = \frac{1}{2} a^{ij} (H_{ijk} - a_{ij|k}^\circ - A_k^h a_{ij|k}^\circ), \\ ^*\theta_{hj}^i D_{ik}^h = \frac{1}{2} a^{ij} (K_{ijk} - a_{ij|k}^\circ), \end{cases}$$

quantities which satisfy the compatibility conditions.

The general solutions of the written equations are

$$(1.6) \quad \begin{cases} B_{jk}^i = \frac{1}{2} a^{ij} (H_{ijk} - a_{ij|k}^\circ - A_k^h a_{ij|k}^\circ) - \theta_{hj}^i U_{ik}^h, \\ D_{jk}^i = \frac{1}{2} a^{ij} (K_{ijk} - a_{ij|k}^\circ) - \theta_{hj}^i V_{ik}^h, \end{cases}$$

with A_j^i , U_{ik}^h , V_{ik}^h — arbitrary Finsler tensor fields.

Having in mind (1.6), the relations (1.3) become

$$(1.7) \quad \begin{cases} N_k^i = \overset{0}{N}_k^i - A_k^i, \\ F_{jk}^i = \overset{0}{F}_{jk}^i + \overset{0}{C}_{jm}^i A_k^m + \frac{1}{2} a^{pi} (a_{pj|k}^\circ + A_k^h a_{pj|k}^\circ - H_{pj|k}) + \theta_{qj}^{ip} U_{pk}^q, \\ C_{jk}^i = \overset{0}{C}_{jk}^i + \frac{1}{2} a^{pi} (a_{pj|k}^\circ - K_{pj|k}) + \theta_{qj}^{ip} V_{pk}^q \end{cases}$$

and we have

Theorem 1.1. *The set of all general almost symplectical Finsler connections $(N_k^i, F_{jk}^i, C_{jk}^i)$ is given by (1.7), where :*

- a) $F\overset{0}{\Gamma} = (\overset{0}{N}_k^i, \overset{0}{F}_{jk}^i, \overset{0}{C}_{jk}^i)$ is a fixed Finsler connection ;

b) $\dot{[}$, $\dot{]}$ denotes the covariant h -derivative and the covariant v -derivative in respect to the fixed connection $F\Gamma^0$;

c) A_k^i , U_{pk}^a , V_{pk}^a are arbitrary Finsler tensor fields.

Remarks. 1. As compared to the formulae (2.8) obtained in [1], the present results (1.7) enjoy the advantage that they do not contain the usual derivative $\partial a_{ij}/\partial y^k$, but only the covariant derivative in respect to the fixed connection.

2. If $H_{ijk}=0$ and $K_{ijk}=0$ we get the set of almost symplectical Finsler connections studied in [2].

3. If $H_{ijk}=2a_{ij}\omega_k$ and $K_{ijk}=2a_{ij}\lambda_k$, where $\omega_k=\omega_k(x, y)$ and $\lambda_k=\lambda_k(x, y)$ are covariant Finsler vectors, from (1.7) we get now the set of conformal almost symplectical Finsler connections, discussed in [3].

2. Among the general almost symplectical Finsler connections (1.7), two are more important.

The first is the general almost symplectical Finsler connections resulting from (1.7) when $A_j^i=0$, $U_{pk}^a=0$, $V_{pk}^a=0$. We shall call them the Kawaguchi general almost symplectical connection [6] associated to $F\Gamma^0$; if denoted $K\Gamma^0=(\overset{K}{N}, \overset{K}{F}, \overset{K}{C})$, then

$$(2.1) \quad \begin{cases} \overset{K}{N}_k^i = \overset{0}{N}_k^i, \\ \overset{K}{F}_{jk}^i = \overset{0}{F}_{jk}^i + \frac{1}{2} a^{pi} (a_{pj|k} - H_{pjk}), \\ \overset{K}{C}_{jk}^i = \overset{0}{C}_{jk}^i + \frac{1}{2} a^{pi} (a_{pj|k} - K_{pjk}). \end{cases}$$

The second remarkable connection may be obtained from (1.7) if we consider as fixed connection $F\Gamma^0$, the Matsumoto connection $M\Gamma=(\overset{m}{N}, \overset{m}{F}, \overset{m}{C})$ and also $A_k^i=0$, $U_{pk}^a=0$, $V_{pk}^a=0$. We call it the general canonical almost symplectical connection; if denoted by $F\Gamma^q=(\overset{q}{N}, \overset{q}{F}, \overset{q}{C})$ it results

$$(2.2) \quad \begin{cases} \overset{q}{N}_k^i = \overset{m}{N}_k^i, \\ \overset{q}{F}_{jk}^i = \overset{m}{F}_{jk}^i + \frac{1}{2} a^{pi} (a_{pj|k}^m - H_{pjk}^m), \\ \overset{q}{C}_{jk}^i = \overset{m}{C}_{jk}^i + \frac{1}{2} a^{pi} (a_{pj|k}^m - K_{pjk}^m). \end{cases}$$

The Matsumoto connection $M\Gamma=(\overset{m}{N}, \overset{m}{F}, \overset{m}{C})$ is given by [6],

$$(2.3) \quad \overset{m}{N}_k^i = \overset{0}{N}_k^i, \quad \overset{m}{F}_{jk}^i = \frac{\partial \overset{0}{N}_j^i}{\partial y^k}, \quad \overset{m}{C}_{jk}^i = \overset{0}{C}_{jk}^i,$$

where N_k^0 is a nonlinear connection, C_{jk}^0 is an arbitrary but fixed connection (in particular $C_{jk}^0=0$) and $\overset{m}{|}, \overset{m}{|}$ means the covariant h -derivative and τ -derivative.

Let $\mathcal{K}$ be the class of $F\overset{0}{\Gamma}$ connections with the property that Kawaguchi general almost symplectical connection $K\overset{0}{\Gamma}$ coincides with the general canonical almost symplectical connection $F\overset{q}{\Gamma}$, i.e.

$$\mathcal{K} = \{F\overset{0}{\Gamma} | K\overset{0}{\Gamma} = F\overset{q}{\Gamma}\}.$$

This class is given by

Theorem 2.1. *All the $\mathcal{K}$ -class connections are*

$$(2.4) \quad \begin{cases} N_k^i = N_k^i, \\ F_{jk}^0 = F_{jk}^i + \frac{1}{2} a^{pi}(a_{pj|k} - H_{pjl}) - {}^* \Theta_{hj}^{im} X_{mk}^h, \\ C_{jk}^0 = C_{jk}^i + \frac{1}{2} a^{pi}(a_{pj|l} - K_{pjl}) - {}^* \Theta_{hj}^{im} Y_{mk}^h, \end{cases}$$

where $M\overset{m}{\Gamma} = (N, \overset{m}{F}, \overset{m}{C})$ is Matsumoto connection and X_{mk}^h, Y_{mk}^h are arbitrary tensor fields.

Proof. Let $A_k^i, F_{jk}^i, C_{jk}^i$ be the difference tensors obtained from the pair $(F\overset{0}{\Gamma}, F\overset{q}{\Gamma})$:

$$(2.5) \quad N_k^i = N_k^i - A_k^i, \quad F_{jk}^0 = F_{jk}^i + C_{jm}^i A_k^m - B_{jk}^i, \quad C_{jk}^0 = C_{jk}^i - D_{jk}^i.$$

Introduce (2.5) in (2.1) and take into account the expressions of the covariant derivatives of a_{ij} in respect to the connections $F\overset{0}{\Gamma}$ and $F\overset{q}{\Gamma}$. It follows

$$(2.6) \quad \begin{cases} N_k^i = N_k^i - A_k^i, \\ F_{jk}^i = F_{jk}^i + C_{jm}^i A_k^m + \frac{1}{2} a^{pi} A_k^h K_{pjh} - {}^* \Theta_{hj}^{ip} B_{pk}^h, \\ C_{jk}^i = C_{jk}^i - A_{hj}^{ip} D_{pk}^h. \end{cases}$$

The condition $K\overset{0}{\Gamma} = F\overset{q}{\Gamma}$ gives the following system of equations

$$(2.7) \quad A_k^i = 0, \quad \Theta_{hj}^{ip} B_{pk}^h = 0, \quad \Theta_{hj}^{ip} D_{pk}^h = 0,$$

with the general solution

$$(2.8) \quad A_k^i = 0, \quad {}^* \Theta_{hj}^{im} X_{mk}^h = B_{jk}^i, \quad D_{jk}^i = {}^* \Theta_{hj}^{im} Y_{mk}^h,$$

where X_{mk}^h and Y_{mk}^h are arbitrary tensor fields.

Substituting in (2.5) we get

$$(2.9) \quad \begin{cases} \overset{0}{N}_k^i = \overset{q}{N}_k^i, \\ \overset{0}{F}_{jk}^i = \overset{q}{F}_{jk}^i - {}^* \theta_{hj}^{im} X_{mk}^h, \\ \overset{0}{C}_{jk}^i = \overset{q}{C}_{jk}^i - {}^* \theta_{hj}^{im} Y_{mk}^h. \end{cases}$$

Now, having in mind (2.9), the last equations (2.9) results in (2.4).

Remarks. 1. If in (2.4) we consider that $H_{ijk}=0$ and $K_{ijk}=0$, we get the class of Finsler connections $\overset{0}{F}\Gamma$ with the property that the associated Kawaguchi almost symplectical connection coincides with the canonical almost symplectical connection. This class was studied in [4].

2. If in (2.4) we consider now $H_{ijk}=2a_{ij}\omega_k$ and $K_{ijk}=2a_{ij}\lambda_k$, where $\omega_k(x, y)$ and $\lambda_k(x, y)$ are covariant Finsler vectors, we obtain the class of Finsler connections $\overset{0}{F}\Gamma$ with the property that the associated Kawaguchi conformal almost symplectical connection coincides with the canonical almost symplectical connection. This class was studied in [5].

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ASUPRA CONEXIUNILOR FINSLER GENERALE APROAPE SIMPLECTICE

(Rezumat)

Se stabilește o nouă formă a relațiilor care dau conexiunile Finsler generale aproape simplectice și se determină clasa conexiunilor Finsler care au proprietatea că conexiunea generală aproape simplectică Kawaguchi, asociată, coincide cu conexiunea generală aproape simplectică canonică.

ON METRICAL CONNECTIONS IN A SPACE OF LINE ELEMENTS

BY

GH. ANDRICIOAEI

1. The geometry of the space of line elements studies the invariants to the transformations of coordinates [4]:

$$(1.1) \quad \begin{aligned} a) \quad x^{i'} &= x^{i'}(x^1, \dots, x^n), \quad y^{i'} = y^{i'} A_i^{i'}, \quad \left(A_i^{i'} = \frac{\partial x^{i'}}{\partial x^i} \right), \\ b) \quad \bar{y}^\alpha &= \lambda(x) y^\alpha, \quad \alpha = 1, \dots, n, \quad \lambda(x) > 0. \end{aligned}$$

The tensors, the parameters of a linear connection and of a nonlinear one are transformed by (1.1) as follows:

$$(1.2) \quad \begin{aligned} a) \quad T_{j'}^{i'} &= T_{ji}^{i'} A_i^{i'} A_{j'}^j, \\ b) \quad \Gamma_{j'k'}^{i'} &= \Gamma_{jk}^i A_i^{i'} A_{j'}^j A_{k'}^k + A_i^{i'} A_{j'k'}^i, \quad \left(A_{jk}^i = \frac{\partial^2 x^i}{\partial x^j \partial x^k} \right), \\ c) \quad N_{j'}^{i'} &= N_{ji}^{i'} A_i^{i'} A_{j'}^j + A_i^{i'} A_{jk}^i y^{j'k'}. \end{aligned}$$

Let $g_{ij}(x, y)$ be a nondegenerate tensor homogeneous of degree zero with respect to y . From g_{ij} we can determine the function

$$(1.3) \quad L^2(x, y) = g_{ij}(x, y) y^i y^j.$$

The covariant differential in this space has the form

$$(1.4) \quad DX^i = dX^i + \bar{L}_{jk}^i X^j dx^k + \bar{M}_{jk}^i X^j dy^k,$$

where $\bar{L}$, $\bar{M}$ are parameters of the affine connection. It is known [4] that $\bar{L}_{jk}^i$ is transformed by formulas (1.2 b) and $\bar{M}_{jk}^i$ by (1.2 a)

We want to study the metrical connections and to investigate the transformations of this connections following the method of prof. R. Miron used in Finsler spaces [5], [6]. The notations are taken from [1] [3], but with some modifications.

Now we write the differential (1.4) in the form

$$(1.5) \quad DX^i = dX^i + L_{jk}^i X^j dx^k + M_{jk}^i X^j dy^k$$

where following notations and conditions are made:

$$(1.6) \quad \det(\delta_j^i + \bar{M}_{oj}^i) \neq 0, \quad (\delta_k^i + \bar{M}_{ok}^i)J_i^j = \delta_k^j, \quad (\delta_k^i + \bar{M}_{ok}^i)J_j^k = \delta_j^i;$$

$$(1.7) \quad L_{jk}^i = \bar{L}_{jk}^i - \bar{M}_{js}^i J_r^s \bar{L}_{ok}^r, \quad M_{jk}^i = \bar{M}_{jr}^i J_k^r, \quad \bar{L}_{ok}^i = \bar{L}_{jk}^i y^j, \quad \bar{M}_{ok}^i = \bar{M}_{jk}^i y^j.$$

From (1.6) we have $\det(J_i^j) \neq 0$, then there exists $(J^{-1})_k^j$, so that

$$(1.8) \quad J_j^i (J^{-1})_k^j = \delta_k^i, \quad J_j^i (J^{-1})_i^k = \delta_k^j.$$

Using this, we have

$$(1.9) \quad J_k^i = \delta_k^i - M_{ok}^i, \quad \bar{L}_{ok}^i = (J^{-1})_i^s L_{ok}^s;$$

$$(1.10) \quad \bar{M}_{jk}^i = M_{jr}^i (J^{-1})_k^r, \quad \bar{L}_{jk}^i = L_{jk}^i + M_{jr}^i (J^{-1})_s^r L_{ok}^s.$$

Then we obtain

$$(1.11) \quad DX^i = X^i|_k dx^k + X^i|_k D y^k,$$

where the covariant derivatives of first and second kind appear:

$$(1.12) \quad X^i|_k = \delta_k X^i + X^r L_{rk}^i, \quad X^i|_k = \partial_r X^i J_k^r + X^r M_{rk}^i$$

and where the following notations are made

$$(1.13) \quad \delta_k = \partial_k - \partial_r N_{rk}^r, \quad N_{rk}^r = L_{rk}^r = L_{jk}^r y^j.$$

Remark. By the transformations (1.1), N_{rk}^r are transformed as parameters of a nonlinear connection, i.e. by formulas (1.2 c).

We mention that besides derivatives (1.12) there exists one more tensorial derivative, given by

$$(1.14) \quad T_{j||r}^i = \partial_r T_j^i.$$

Applying (1.12) to the directional element y^i , we have

$$(1.15) \quad y^i|_k = 0, \quad y^i|_k = \delta_k^i.$$

The determination of the parameters L and M of the metrical connection is made by solving the equations

$$(1.16) \quad g_{ij|k} = \delta_i g_{ij} - L_{ijk} - L_{jik} = 0; \quad g_{ij|k} = g_{ij||r} J_k^r - M_{ijk} - M_{jik} = 0.$$

This problem is completely solved and L and M are obtained in a general form in [3] (Eqs. (3.6) and (4.1)).

In the following we shall use the Obata's operators [8] defined in the space of line elements by

$$(1.17) \quad \Theta_{kl}^{ij} = \frac{1}{2}(\delta_k^i \delta_l^j - g^{ij} g_{kl}); \quad * \Theta_{kl}^{ij} = \frac{1}{2}(\delta_k^i \delta_l^j + g^{ij} g_{kl})$$

which operates over some tensor T as

$$(1.18) \quad \Theta_{kl}^{ij} T_{...}^{...}, \quad * \Theta_{kl}^{ij} T_{...}^{...}$$

and for that are verified the relations

$$(1.19) \quad \Theta + {}^*\Theta = \delta\delta, \quad \Theta.\Theta = \Theta, \quad \Theta.{}^*\Theta = 0, \quad {}^*\Theta.\Theta = 0, \quad {}^*\Theta.{}^*\Theta = {}^*\Theta.$$

In this space we have

L e m m a. A tensor V being given, the linear equations

$$(1.20) \quad \Theta T = V \quad (\text{respectively } {}^*\Theta T = V)$$

with unknown tensor T admit a solution if and only if V satisfies

$$(1.21) \quad {}^*\Theta V = 0 \quad (\text{respectively } \Theta V = 0)$$

and then the general solution is given by

$$(1.22) \quad T = V + {}^*\Theta W \quad (\text{respectively } T = V + \Theta W),$$

W being an arbitrary tensor of the same type as T .

The proof of the Lemma is identical to that given in [6].

2. Now we denote

$$(2.1) \quad G_{ij} = g_{ij} + \frac{1}{2} g_{kjl} y^k$$

and we suppose that

$$(2.2) \quad \det (G_{ij}) \neq 0.$$

Then there exists G^{ji} such as

$$(2.3) \quad G_{ij} G^{ji} = \delta_i^j.$$

Definition 2.1. A metrical line element space in which (2.2) is true, is called a *special line elements space*.

When we shall have to deal with a given connection in the space of line elements, we shall refer to the set of parameters $(N_j^i, J_j^i, L_{jk}^i, M_{jk}^i)$ or shortly (N, J, L, M) . It is easy to verify that a particular solution of (1.16) is the following

$$(2.4) \quad \begin{aligned} \dot{L}_{ijk} &= g_{jk} \Gamma_{ik}^h + \frac{1}{2} \left(g_{ij:k} - \frac{y_i}{L^2} y_{j:k} + \frac{y_j}{L^2} y_{i:k} \right), \\ \dot{M}_{ijk} &= \frac{1}{2} \left(g_{ij||k} - \frac{y_i}{L^2} g_{tj||k} y^t + \frac{y_j}{L^2} g_{ti||k} y^t \right), \end{aligned}$$

where $L_{ijk} = g_{jl} L_{ik}^l$, $M_{ijk} = g_{jl} M_{ik}^l$ and (see [3])

$$\Gamma_{jk}^i = \frac{1}{2} \gamma^{ih} (\partial_k \gamma_{jh} + \partial_j \gamma_{kh} - \partial_h \gamma_{jk} - \gamma_{jhl} \Gamma_{ik}^l - \gamma_{khl} \Gamma_{ij}^l + \gamma_{jlk} \Gamma_{ih}^l),$$

$$G^i = \frac{1}{2} \gamma^{ih} \left(\frac{\partial^2 L^2}{\partial y^h \partial x^m} y^m - \frac{\partial L^2}{\partial x^h} \right); \quad \gamma_{ij} = \frac{1}{2} \frac{\partial^2 L^2}{\partial y^i \partial y^j},$$

$$Y_{i;k} = \partial_k Y_i - Y_{i||r} \Gamma_{sk}^r y^s - Y_r \Gamma_{ik}^r.$$

Now we propose to determine the connections for which we have

$$(2.5) \quad N_k^i = \dot{N}_k^i + A_k^i, \quad J_k^i = \dot{J}_k^i + H_k^i, \quad L_{jk}^i = \dot{L}_{jk}^i + B_{jk}^i, \quad M_{jk}^i = \dot{M}_{jk}^i + D_{jk}^i,$$

where A_k^i , B_{jk}^i , H_k^i , D_{jk}^i are some arbitrary tensors with homogeneity of degree 1, 0, 0 and -1 respectively with respect to y . This means that it must verify the relations

$$(2.6) \quad \begin{aligned} \hat{c}_k g_{ij} - g_{ij||r} (\dot{N}'_k + A'_k) - g_{rj} (\dot{L}'_{ik} + B'_{ik}) - g_{ir} (\dot{L}'_{jk} + B'_{jk}) &= 0, \\ g_{ij||r} (\dot{J}'_k + H'_k) - g_{rj} (\dot{M}'_{ik} + D'_{ik}) - g_{ir} (\dot{M}'_{jk} + D'_{jk}) &= 0. \end{aligned}$$

Because the connection $(\dot{N}, \dot{J}, \dot{L}, \dot{M})$ is metrical one, (2.6) becomes

$$(2.7) \quad g_{rj} B'_{ik} + g_{ir} B'_{jk} = -g_{ij||r} A'_k, \quad g_{rj} D'_{ik} + g_{ir} D'_{jk} = g_{ij||r} H'_k,$$

which is equivalent to

$$(2.8) \quad B'_{ik} = -\frac{1}{2} g^{mj} g_{ij||r} A'_k + \Theta_{is}^{rm} W_{rk}^s, \quad D'_{ik} = \frac{1}{2} g^{mj} g_{ij||r} H'_k + \Theta_{is}^{rm} Z_{rk}^s.$$

From the relations (2.5), (1.9) and (1.13) we have

$$(2.9) \quad A'_k = B'_{ok} = B'_{jk} y^j, \quad H'_k = -D'_{ok} = -D'_{jk} y^j.$$

From (2.8) and (2.9) we obtain

$$(2.10) \quad A'_k (g_{rl} + \frac{1}{2} g_{l||r} y^l) = \Theta_{is}^{rm} W_{rk}^s g_{ml}.$$

Now, by using (2.1), from (2.10) it results

$$(2.11) \quad A'_k = \frac{1}{2} (M_{ok} - W_{ok}) G^{lj}, \quad W_{ijk} = g_{ir} W'_{rk}.$$

By a similar way, we obtain for H'_k the expression

$$(2.12) \quad H'_k = -\frac{1}{2} (Z_{ok} - Z_{ok}) G^{lj}, \quad Z_{ijk} = g_{ir} Z'_{rk}.$$

Replacing (2.11), (2.12) and (2.8) in (2.5) we obtain

$$(2.13) \quad \begin{aligned} N'_k &= \dot{N}'_k + \frac{1}{4} (W_{ok} - W_{ok}) G^{li}, \quad J'_k = \dot{J}'_k - \frac{1}{2} (Z_{ok} - Z_{ok}) G^{li}, \\ L'_{jk} &= \dot{L}'_{jk} - \frac{1}{4} g^{is} g_{js||r} (W_{ok} - W_{ok}) G^{lr} + \Theta_{js}^{ri} W_{rk}^s, \\ M'_{jk} &= \dot{M}'_{jk} - \frac{1}{4} g^{is} g_{js||r} (Z_{ok} - Z_{ok}) G^{lr} + \Theta_{js}^{ri} Z_{rk}^s. \end{aligned}$$

Thus we have

Theorem 2.1. *If $(\dot{N}, \dot{J}, \dot{L}, \dot{M})$ is a fixed metrical connection in a special line elements space and W_{rk}^s, Z_{rk}^s are arbitrary tensors with homogeneity of degree 0 and -1 respectively with respect to y , then the connections (N, J, L, M) given by (2.13) are some general connections in this space.*

Now, if in (2.13) we take $\Theta_{js}^{ri} W_{rk}^s = P_{jk}^i$, $\Theta_{js}^{ri} Z_{rk}^s = Q_{jk}^i$, then it results $\frac{1}{2} (W_{jlk} - W_{ljk}) = P_{jlk}$, $\frac{1}{2} (Z_{jlk} - Z_{ljk}) = Q_{jlk}$, where $P_{jlk} = P_{jk}^i g_{il}$ and $Q_{jlk} = Q_{jk}^i g_{il}$. In view of (1.19) we have $*\Theta \cdot P = 0$ and $*\Theta \cdot Q = 0$ which proves that P_{jk}^i and Q_{jk}^i are hibrid tensors in i and j .

The relations (2.13) are written now in the form

$$(2.14) \quad \begin{aligned} N'_k &= \dot{N}'_k + P_{ok} G^{li}, \quad J'_k = \dot{J}'_k - Q_{ok} G^{li}, \\ L'_{jk} &= \dot{L}'_{jk} - \frac{1}{2} g^{is} g_{js||r} P_{ok} G^{lr} + P_{jk}^i, \\ M'_{jk} &= \dot{M}'_{jk} - \frac{1}{2} g^{is} g_{js||r} Q_{ok} G^{lr} + Q_{jk}^i. \end{aligned}$$

Then we have

Theorem 2.2. *If $(\overset{\circ}{N}, \overset{\circ}{J}, \overset{\circ}{L}, \overset{\circ}{M})$ is the metrical connection (2.4) in a special line elements space and P_{jk}^i, Q_{jk}^i are arbitrary hybrid in i, j and homogeneous of degree 0 and -1 in respect to y respectively tensors, then the connections (N, J, L, M) given by (2.14) are some general connections in this space.*

Let (N, J, L, M) , $(\bar{N}, \bar{J}, \bar{L}, \bar{M})$ be any two general metrical connections given by (2.14) and let P_{jk}^i, Q_{jk}^i be arbitrary hybrid in i, j and homogeneous of degree 0, -1 respectively tensors. Then the two connections are related by

$$(2.15) \quad \begin{aligned} \bar{N}_k^i &= N_k^i + P_{0lk} G^{li}, & \bar{J}_k^i &= J_k^i - Q_{0lk} G^{li}, \\ \bar{L}_{jk}^i &= L_{jk}^i - \frac{1}{2} g^{is} g_{js||r} P_{0lk} G^{lr} + P_{jk}^i, \\ \bar{M}_{jk}^i &= M_{jk}^i - \frac{1}{2} g^{is} g_{js||r} Q_{0lk} G^{lr} + Q_{jk}^i, \end{aligned}$$

which gives a set of transformations of connections in the special line elements space. It is easy to show

Theorem 2.3. *The set of all transformations (2.15) together with the mapping $(N, J, L, M) \rightarrow (\bar{N}, \bar{J}, \bar{L}, \bar{M})$ is a commutative group.*

Applying the operator ${}^*\Theta$ to (2.15) we obtain the relations

$$(2.16) \quad \begin{aligned} {}^*\Theta_{pi}^{jm} \bar{L}_{jk}^i + \frac{1}{2} g^{ms} g_{ps||r} \bar{N}_k^r &= {}^*\Theta_{pi}^{jm} L_{jk}^i + \frac{1}{2} g^{ms} g_{ps||r} N_k^r, \\ {}^*\Theta_{pi}^{jm} \bar{M}_{jk}^i - \frac{1}{2} g^{ms} g_{ps||r} \bar{J}_k^r &= {}^*\Theta_{pi}^{jm} M_{jk}^i - \frac{1}{2} g^{ms} g_{ps||r} J_k^r, \end{aligned}$$

from which we have

Theorem 2.4. *The geometric objects*

$$(2.17) \quad \mathcal{L}_{pk}^m = {}^*\Theta_{pi}^{jm} L_{jk}^i + \frac{1}{2} g^{ms} g_{ps||r} N_k^r, \quad \mathcal{M}_{pk}^m = {}^*\Theta_{pi}^{jm} M_{jk}^i - \frac{1}{2} g^{ms} g_{ps||r} J_k^r$$

are invariants of the transformations (2.15).

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ASUPRA CONEXIUNILOR METRICE ÎN SPAȚIUL DE ELEMENTE LINIARE

(Rezumat)

Plecînd de la lucrările [1], [2], [3] se studiază conexiunile metrice în spațiul de elemente liniare și se determină un grup de transformări ale acestor conexiuni precum și doi invarianti ai acestui grup.

ON THE APPROXIMATION OF THE MOMENTS OF CERTAIN RANDOM VARIABLES

BY

ILIE BURDUJAN

1. Random variables — the probability density functions of which being solutions of some partial differential equations, for ex. the Boltzmann's equation — are used in the solving of certain problems regarding plasma physics. It is known enough that in addition to other numerical characteristics, moments of various orders are being introduced in the study of these variables. The effective determination of these moments is conditioned by the also effective knowledge of the probability density functions of these variables. Since the accurate determination of the probability density functions is not always possible, a numerical approximation of the values of these functions, or an approximation by trigonometric polynomials may be used. It is also possible to pass to more simple differential equations, to equations containing finite differences, or to such equations containing finite differences and derivatives of the image function as well by making use of the Laplace transformation. But the use of the Laplace transformation is becoming more difficult due to the fact that those results concerning the image function are presenting difficulties when interpreted.

The problem of finding of an approximation for the various moments of a random variable, with the help of certain expressions which are using the values of the image function of probability density function within the points of a nets is arised. The sugestion for the solving of this problem is comming from the Efragmén's theorem (see [2] p. 73), which we have used in the proof of Theorem 1.

2. We shall only deal with the special class of original function $f: \mathbb{R}_+ \rightarrow \mathbb{R}$ for which two real numbers $M > 0$ and $s \geq 0$ exist, such that $|f(t)| \leq Me^{st}$. The real number s is called the index of f .

We prove first

Lemma 1. *Lets $f: \mathbb{R}_+ \rightarrow \mathbb{R}$ be an original function with the index s and $F(p)$ his image function by Laplace transformation. Then the series*

$$(1) \quad \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} e^{kxt} F(kx)$$

absolutely converges, whenever $x > s$.

Proof. It is well known (see [3], p. 131) that $F(kx) \rightarrow 0$ for $k \rightarrow \infty$. Then there exists a positive integer N_ε such that for every $k > N_\varepsilon$ we have $|F(kx)| < \varepsilon$. Since

$$\left| \frac{(-1)^{k-1}}{k!} e^{kxt} F(kx) \right| < \varepsilon \frac{e^{kxt}}{k!}$$

whenever $k > N_\varepsilon$, and $\sum_{k=0}^{\infty} \frac{e^{kxt}}{k!} = e^{xt}$ the result is that the series (1) absolutely converges for $x > s$.

Lemma 2. Under the assumptions of the Lemma 1, there is a real positive number T_ε such that

$$(2) \quad \left| \int_0^{+\infty} e^{-kxu} f(u) du - \int_0^{T_\varepsilon} e^{-kxu} f(u) du \right| < \varepsilon$$

whenever $x > s + 1$.

Proof. If $T > \ln \frac{M}{\varepsilon}$, we have

$$\left| \int_T^{+\infty} e^{-kxu} f(u) du \right| \leq \frac{M}{kx-s} \frac{1}{e^{(kx-s)T}} < \varepsilon.$$

Therefore, for $T_\varepsilon = \ln \frac{M}{\varepsilon} + 1$ we have (2).

Observations 1. The series (1) absolutely converges for every original function $f: \mathbb{R}_+ \rightarrow \mathbb{C}$ such that $F(kp)$ exists for any $k \in \mathbb{N}$.

2. The statement of Lemma 2 is true for any original function $f: \mathbb{R}_+ \rightarrow \mathbb{C}$ if the real number x is changed by a complex number p for which $\operatorname{Re} p > s + 1$.

3. The real number T_ε defined by Lemma 2 is not depending on $x > s + 1$ or $t \in \mathbb{R}_+$.

We have

Theorem 1. Let $f: \mathbb{R}_+ \rightarrow \mathbb{R}$ be an original function absolutely integrable on $\mathbb{R}_+$, with the index s and $F(p)$ be his Laplace transformation. Then, for every $\varepsilon > 0$, whenever $x > s + 1$ and $t \in \mathbb{R}_+$, we have:

$$(3) \quad \left| \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} e^{kxt} F(kx) - \int_0^t f(u) du \right| < \varepsilon.$$

Proof. In accordance with the Lemma 2 there exists $T_\varepsilon > t > 0$ such that

$$\left| \int_0^{+\infty} e^{-kxu} f(u) du - \int_0^{T_\varepsilon} e^{-kxu} f(u) du \right| < \frac{\varepsilon}{4} \quad \text{for } x > s + 1.$$

Applying the Phragmén's theorem on $[0, T_\varepsilon]$, the result is that there exists $\alpha_\varepsilon > 0$ such that

$$\left| \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \int_0^{T_\varepsilon} e^{kx(t-u)} f(u) du - \int_0^t f(u) du \right| < \frac{\varepsilon}{4}$$

whenever $x > \max(s+1, \alpha_\varepsilon)$. Because the series (1) and the series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \int_0^{T_\varepsilon} e^{kx(t-u)} f(u) du$$

are respectively convergent, then there exists the positive integers N_ε and N_ε'' such that, for $p > N_\varepsilon = \max(N_\varepsilon', N_\varepsilon'')$, we have

$$\left| \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \int_0^{T_\varepsilon} e^{kx(t-u)} f(u) du - \sum_{k=1}^p \frac{(-1)^{k-1}}{k!} \int_0^{T_\varepsilon} e^{kx(t-u)} f(u) du \right| < \frac{\varepsilon}{4}$$

and

$$\left| \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} e^{kxt} F(kx) - \sum_{k=1}^p \frac{(-1)^{k-1}}{k!} e^{kxt} F(kx) \right| < \frac{\varepsilon}{4},$$

respectively. For the above choice of x , T_ε and N_ε' and whenever $p > N_\varepsilon$ we obtain

$$\left| \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} e^{kxt} F(kx) - \int_0^t f(u) du \right| < \varepsilon$$

if T_ε satisfies, in addition, the condition $\int_{T_\varepsilon}^{+\infty} |f(u)| du < \frac{\varepsilon}{4e}$.

We have, indeed

$$\begin{aligned} & \left| \sum_{k=1}^p \frac{(-1)^{k-1}}{k!} e^{kxt} F(kx) - \sum_{k=1}^p \frac{(-1)^{k-1}}{k!} e^{kxt} \int_0^{T_\varepsilon} e^{-kxu} f(u) du \right| = \\ & = \left| \sum_{k=1}^p \frac{(-1)^{k-1}}{k!} e^{kxt} \int_{T_\varepsilon}^{+\infty} e^{-kxu} f(u) du \right| \leq \sum_{k=1}^p \frac{1}{p!} \int_{T_\varepsilon}^{+\infty} |f(u)| du < \frac{\varepsilon}{4}. \end{aligned}$$

Thus the assertion of the theorem is proved.

Remark. The real number x , defined by the Theorem 1 may be chosen to be a positive integer.

Consequence 1. Under the assumptions of the Theorem 1, for every $\varepsilon > 0$ there exists a positive integer N_ε such that for $p > N_\varepsilon$ we have

$$\left| \sum_{k=1}^p \frac{(-1)^{k-1}}{k!} e^{kxt} F(kx) - \int_0^t f(u) du \right| < \varepsilon.$$

Proof. The statement results immediately from the Lemma 1 and the Theorem 1.

Observation. If we choose the positive integer x by the Theorem 1, when using the Consequence 1, it may result that, for a fixed $t \in T_+$,

$\int_0^t f(u) du$ may be approximated by finite linear combinations having constant coefficients of elements of sequence $\{e^{nt}\}_{n \in \mathbb{N}}$. Therefore, every function $g: [0, T] \rightarrow \mathbb{R}_+$ representable by $\int_0^t f(u) du$, may be approximated by such

linear combinations. Particularly, every derivable function may be approximated in this way.

By using the properties of Laplace transformation (the derivation of the image) the Theorem 1 may be completed in the following

Theorem 1'. Under the assumptions of the Theorem 1, for every $\varepsilon > 0$, there exists $x \in \mathbb{R}_+$ such that

$$\left| \sum_{k=1}^{\infty} \frac{(-1)^{k+s-1}}{k!} e^{kxt} F^{(s)}(kx) - \int_0^t u^s f(u) du \right| < \varepsilon, \quad s=0, 1, 2, \dots$$

where $F^{(3)}(p) = F(p)$.

Particularly, if the function f is the probability density function for random variable, we obtain

Theorem 2. Let $f: \mathbb{R}_+ \rightarrow \mathbb{R}$ be the probability density function for a random variable. Then, there exists $x \in \mathbb{R}_+$ and $t \in \mathbb{R}_+$ such that

$$\left| \sum_{k=1}^{\infty} \frac{(-1)^{k+s-1}}{k!} e^{kxt} F^{(s)}(kx) - \int_0^t u^s f(u) du \right| < \varepsilon, \quad s=0, 1, 2, \dots$$

where $F^{(3)}(p) = F(p)$.

Proof. Let us define $t \in \mathbb{R}_+$ by

$$\int_0^t u^s f(u) du - \int_0^t u^s f(u) du < \frac{\varepsilon}{2}.$$

Then the theorem results immediately from Theorem 1'. Evidently, this choice of t is depending on s .

Consequence 2. Let $f: \mathbb{R}_+ \rightarrow \mathbb{R}$ be the probability density function for a random variable. Then, there exists $x \in \mathbb{R}_+$, $t_0 \in \mathbb{R}_+$ and the positive integer N_ε such that, for any $p > N_\varepsilon$ we have

$$\left| \sum_{k=1}^p \frac{(-1)^{k+s-1}}{k!} e^{kxt} F^{(s)}(kx) - \int_0^t u^s f(u) du \right| < \varepsilon, \quad s=0, 1, 2, \dots$$

Since the choice of T_x is not depending on x and the series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} e^{kxt} F(kx) \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} e^{kxt} \int_0^{T_x} e^{kx(t-u)} f(u) du$$

are respectively uniformly convergent, we have

Theorem 3. *Under the assumptions of the Theorem 1, for $t \in \mathbb{R}_+$ we have*

$$\lim_{x \rightarrow \infty} \sum_{k=1}^{\infty} \frac{(-1)^{k+s-1}}{k!} F^{(s)}(kx) = \int_0^t u^s f(u) du, \quad s=0, 1, 2, \dots$$

where $F^{(0)}(p) = F(p)$.

From the Theorem 2 we have also

Theorem 4. *Let $f: \mathbb{R}_+ \rightarrow \mathbb{R}$ be a probability density function for a random variable. Then, there exists $t_0 \in \mathbb{R}_+$ such that for every $t > t_0$ we have*

$$\lim_{x \rightarrow \infty} \sum_{k=1}^{\infty} \frac{(-1)^{k+s-1}}{k!} e^{kxt} F^{(s)}(kx) = \int_0^{+\infty} u^s f(u) du, \quad s=0, 1, 2, \dots$$

By using the Consequence 2, the moments of order s , for the random variable with the probability density function $f: \mathbb{R}_+ \rightarrow \mathbb{R}$, may be approximated by the sum

$$\sum_{k=1}^n \frac{(-1)^{k+s-1}}{k!} e^{kxt} F^{(s)}(kx).$$

The order of the error of this sum is that of the term $\frac{e^{(k+1)xt}}{(k+1)!} F^{(s)}((k+1)x)$.

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ASUPRA APROXIMĂRII MOMENTELOR UNOR ANUMITE VARIABLE ALEATOARE

(Rezumat)

Se propune un procedeu de aproximare a momentelor unor variabile aleatoare ale căror densități de probabilitate sînt soluții ale unor ecuații diferențiale ce nu pot fi determinate efectiv, folosind sumele parțiale ale unor serii alternante ce conțin valorile în nodurile unei rețele ale transformatelor Laplace ale densităților de probabilitate.

PROIECȚIA CILINDRICĂ PE BISECTORUL AL DOILEA

DE

I. RUSU, MARIANA VASILACHE, CARMEN MARICA și CORNELIA EFTIMESCU

1. Introducere. Planele bisectoare sînt plane axiale care împart dihedrele ortogonale în cîte două părți egale. În epura Monge, proiecția orizontală a punctelor bisectorului $[B_2]$ se confundă cu proiecția lor verticală.

Construcția proiecției unui obiect pe planul bisector $[B_2]$ se poate realiza cu ajutorul punctelor proiecției, ca intersecții dintre proiectantele considerate și planul $[B_2]$.

Dar intersecția unei drepte cu bisectorul al doilea este un punct, care în sistemul Monge are proiecțiile confundate, ceea ce face ca el să se găsească la intersecția proiecțiilor dreptei (proiectantei) considerate. Această observație asociată cu proprietăți cunoscute ale proiecției paralele permite soluționarea simplă a unor probleme. De pildă cum este următoarea: să se reprezinte pe o dreaptă de profil un punct, cunoscînd una din cele două proiecții, dar fără utilizarea planului lateral de proiecție.

În fig. 4 s-a reprezentat dreapta de profil prin punctele A și B în dublă proiecție ortogonală iar pe $(a'b')$ s-a considerat proiecția c' . Pentru a determina proiecția orizontală c a punctului C s-au dus prin a' ; b' și c' proiecțiile verticale ale proiectantelor de o direcție arbitrară, deci paralele între ele. Apoi prin a și b s-au dus proiecțiile orizontale ale acestor proiectante care se intersectează cu primele în α și respectiv în β . Dreapta $(\alpha\beta)$ este intersectată în γ de proiectanta din c' . Prin urmare dacă din acest punct γ se duce o proiectantă paralelă la $(\alpha\alpha)$ sau $(\beta\beta)$, ea intersectează proiecția (ab) în c .

Rezultă că dreapta $(\alpha\beta)$ este de fapt proiecția paralelă pe $[B_2]$ a dreptei de profil (AB) .

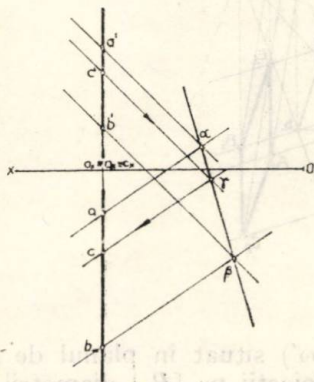


Fig. 1

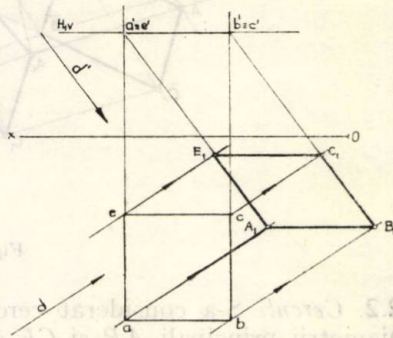


Fig. 2

2. Proiecția paralelă a figurilor plane. 2.1. Pătratul. Se consideră ca exemplu pătratul $ABCE$ situat în planul de nivel $[H_1]$, (fig. 2). Intersectând proiectanta fiecărui vîrf cu bisectorul doi, se găsesc vîrfurile A_1 ; B_1 ; C_1 și E_1 , care determină paralelogramul proiecției. În acest scop se consideră direcția de proiectare (D) (d, d') pentru ca să se poată trasa în epură proiecțiile proiectantelor fiecărui vîrf. Din intersecția proiecțiilor fiecărei proiectante corespunzătoare unui anumit vîrf rezultă vîrfurile A_1 ; B_1 ; C_1 și E_1 .

Proiecția paralelă a unui poligon plan, de exemplu tot un pătrat, situat într-un plan de capăt pe planul bisector doi $[B_2]$ este rezolvată în epura din fig. 3. Astfel, în planul de capăt dat prin urmele sale este reprezentat pătratul $ABCE$ în care diagonală AB ($ab, a'b'$) este frontală iar diagonală CE ($ce, c'e'$) este dreaptă de capăt.

S-au considerat direcțiile de proiectare (D) (d, d'), (D_1) (d_1, d'_1) și (D_2) (d_2, d'_2), fiecareia corespunzându-i o proiecție a aceluiași poligon. Astfel pentru direcția de proiectare $\bar{D}(\bar{d}, \bar{d}')$ corespunde proiecția $ABCE$ care este un paralelogram. Punctul proiecție pe planul $[B_2]$ s-a găsit la intersecția proiecțiilor (d_0) și (d'_0) corespunzătoare proiectantei (D_0) ce trece prin vîrfurile A . Se observă că ea intersectează axa de afinitate Δ (δ, δ') = $[P] \cap [B_2]$ în punctul α , iar latura (bc) în β . Vîrfurile A_1 și A_2 ale altor proiecții în B_2 ale aceluiași pătrat sînt: $A_1 = (D_{01}) \cap [B_2]$ și $A_2 = (D_{02}) \cap [B_2]$. Practic, în epură $A_1 = (d_{01}) \cap (d'_{01})$ și $A_2 = (d_{02}) \cap (d'_{02})$.

Celelalte vîrfuri ale proiecțiilor se obțin prin corespondența de afinitate ce există între poligonul (pătratul) $ABCE$ dar și $\bar{ABCE}$; $A_1B_1C_1E_1$; $A_2B_2C_2E_2$ ca proiecții afine ale acestuia.

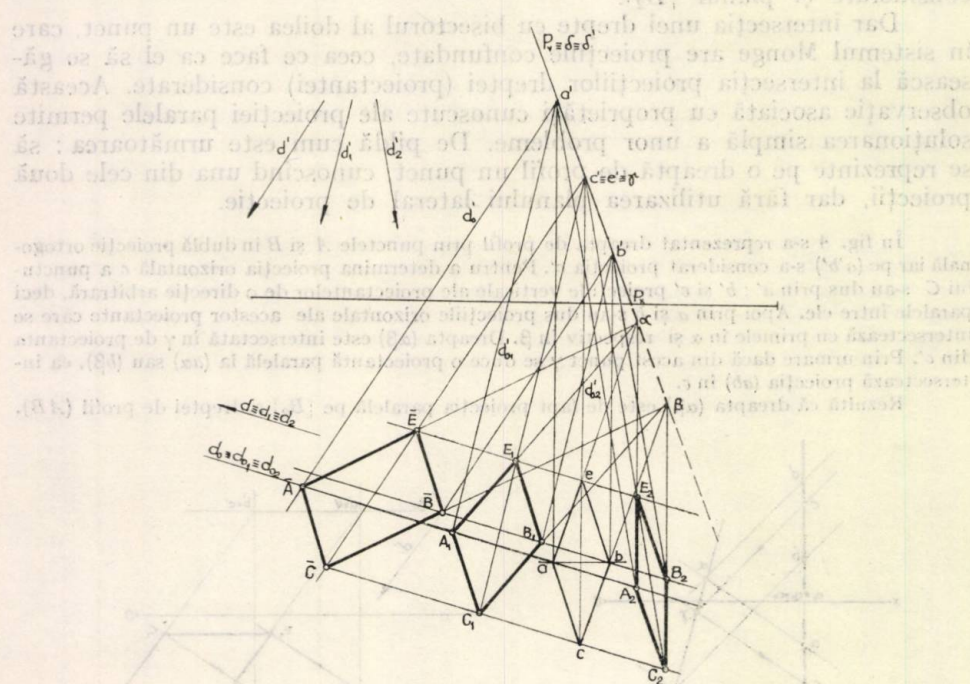


Fig. 3

2.2. Cercul. S-a considerat cercul $\Omega(\omega, \omega')$ situat în planul de front $[F]$. Diametrii principale AB și CE au ca proiecții pe $[B_2]$ diametrii con-

jugați A_1B_1 și C_1E_1 . Extremitățile acestora s-au găsit ca proiecții pe planul $[B_2]$ ale punctelor A ; B ; C și E de pe cercul Ω (fig. 4). Astfel, prin punctele $A(a, a')$; $B(b, b')$; $C(c, c')$ și $E(e, e')$ s-au dus proiectante paralele cu $D(d, d')$, ale căror proiecții se intersectează în $A_1(a_1 \equiv a'_1)$; $B_1(b_1 \equiv b'_1)$; $C_1(c_1 \equiv c'_1)$ și $E_1(e_1 \equiv e'_1)$, puncte care determină diametrii conjugați A_1B_1 și C_1E_1 . Intersecția lor este punctul Ω_1 ca proiecție pe planul $[B_2]$ a centrului cercului Ω .

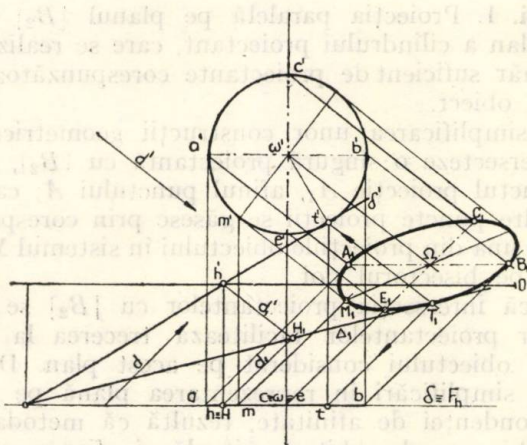


Fig. 4

Problema propusă se completează cu construcția tangentei în punctul T_1 de pe elipsa Ω_1 . În acest scop au fost găsite proiecțiile t și t' în sistemul Ox ale punctului T de pe cercul dat. Tangenta în T la cercul Ω este frontala $\Delta(\delta, \delta')$, căreia îi corespunde urma orizontală $(H)(h, h')$. Acest punct proiectat cu proiectanta $(D_1)(d_1, d') \parallel (D)(d, d')$ pe planul $[B_2]$ are ca proiecție punctul H_1 . Se observă că dreapta $H_1T_1 \equiv \Delta_1$ este proiecția tangentei Δ și deci tangentă în T_1 la elipsa proiecției

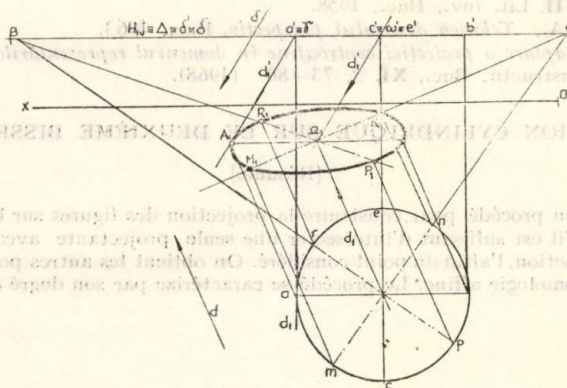


Fig. 5

În fig. 5 s-a considerat cercul $\Omega(\omega, \omega')$ situat în planul de nivel $[H_1]$, reprezentat în epură prin urma sa verticală (H_{1v}) . Elipsa proiecție Ω_1 este privită drept curba afină a cercului Ω dat. În cerc s-au considerat și diametrii principali MN și PR ale căror prelungiri întâlnesc axa de

afinitate $\Delta(\delta, \delta') = [H_1] \cap [B_2]$ în punctele α și β . Se duce proiectanta $(D_1) (d_1, d'_1) \parallel (D) (d, d')$ prin centrul Ω , pe care îl proiectează pe planul $[B_2]$ în Ω_1 . Acest punct împreună cu α și β determină dreptele $\alpha\Omega_1$ și $\beta\Omega_1$ care prin intersecția cu proiecțiile orizontale ale proiectantelor punctelor extremități menționate, conduc la construcția diametrilor M_1N_1 și P_1R_1 afini față de diametrii principali MN și PR din cercul Ω , ceea ce permite trasarea elipsei Ω_1 .

În epură s-a găsit proiecția A_1 pe planul $[B_2]$ a punctului $A (a, a')$ de pe același cerc. Punctul γ de pe axa $(\Delta) \equiv (H_{1v})$ împreună cu A_1 determină tangenta în acest punct la elipsa proiectie Ω_1 .

3. Concluzii. 1. Proiecția paralelă pe planul $[B_2]$ înseamnă secționarea cu acest plan a cilindrului proiectant, care se realizează intersectînd cu $[B_2]$ un număr suficient de proiectante corespunzătoare punctelor caracteristice unui obiect.

2. Pentru simplificarea unor construcții geometrice plane este suficient să se intersecteze o singură proiectantă cu $[B_2]$, din care rezultă, de exemplu, punctul proiecție A_1 , afinul punctului A , caracteristic obiectului dat. Celelalte puncte proiecții se găsesc prin corespondența de omologie afină dintre una din proiecțiile obiectului în sistemul Monge și proiecția aceluiași obiect pe bisectorul doi.

3. Faptul că intersecția proiectantelor cu $[B_2]$ se reduce la intersecția proiecțiilor proiectantelor facilitează trecerea la construcția imaginii (proiecției) obiectului considerat pe acest plan. Dacă la aceasta se adaugă și unele simplificări în reprezentarea plană pe planul $[B_2]$ prin utilizarea corespondenței de afinitate, rezultă că metoda indicată de reprezentare pe $[B_2]$ este deosebit de simplă și eficientă.

Primită la 22 mai 1978

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LÀ PROJECTION CYLINDRIQUE SUR LE DEUXIÈME BISSECTEUR $[B_2]$

(Résumé)

On propose un procédé pour construire la projection des figures sur le deuxième bissecteur. On montre qu'il est suffisant d'intersecter une seule projectante avec $[B_2]$, d'où il en résulte le point projection, l'afin du point considéré. On obtient les autres points projection par correspondance d'homologie affine. Le procédé se caractérise par son degré de simplicité.

TORSION OF THE ISOTROPIC AND HOMOGENEOUS MICROPOLAR ELASTIC NATURALLY TWISTED BEAMS

BY

I. CRĂCIUN

1. Introduction. The theory of naturally twisted beams in the linear theory of elasticity was initiated and developed by C.I. Borș in the book [1] and in various papers (see e.g. [2]–[5]).

This paper is concerned with the torsion problem of the naturally twisted beams composed of isotropic and homogeneous micropolar elastic materials. This problem is reduced to a general problem for a cylindrical beam [6], [9], [10].

A naturally twisted beam is a body $\mathcal{B}$ limited by the planes $x_3=0$, $x_3=h$ ($h>0$) and by a surface $\mathcal{F}$, its equation being given by

$$(1) \quad f(x_1 - kx_2x_3, x_2 + kx_1x_3) = 0,$$

where k is a small parameter such that the square and higher powers of k can be neglected by comparison with k .

The axes of the rectangular Cartesian coordinate system $Ox_1x_2x_3$, used here, are taken as in [10]–[12].

The cross-section of the beam lies in the plane $x_3=p$, $0 \leq p \leq h$, is denoted by S_p and is a simply-connected-region.

We shall employ the usual summation and differentiation conventions: Greek subscripts are understood to range over the integers (1, 2), whereas Latin subscripts to the range (1, 2, 3); summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding coordinate.

2. Basic Equations. We assume that the body force and the body couples are absent. The basic equations in the static theory of homogeneous and isotropic micropolar elastic solids are [7], [8]:

i) the equilibrium equations

$$(2) \quad \frac{\partial t_{ji}}{\partial x_j} = 0, \quad \frac{\partial m_{ij}}{\partial x_j} + \epsilon_{imn} t_{mn} = 0, \quad \text{in } \mathcal{B};$$

ii) the constitutive relations

$$(3) \quad \begin{cases} t_{ij} = \lambda \varepsilon_{rr} \delta_{ij} + (\mu + \nu) \mu_{ij} + \mu \varepsilon_{ji}, \\ m_{ij} = \alpha \frac{\partial \varphi_r}{\partial x_r} \delta_{ij} + \beta \frac{\partial \varphi_i}{\partial x_j} + \gamma \frac{\partial \varphi_j}{\partial x_i}, \end{cases} \text{ in } \mathcal{B} \cup \partial \mathcal{B};$$

iii) the kinematic relations

$$(4) \quad \varepsilon_{ij} = \frac{\partial u_j}{\partial x_i} + \varepsilon_{jim} \varphi_m, \quad \text{in } \mathcal{B} \cup \partial \mathcal{B};$$

iv) the Cauchy's relations

$$(5) \quad t_i = t_j n_j, \quad m_i = m_j n_j, \quad \text{on } \partial \mathcal{B},$$

where $\mathcal{B}$ is the domain occupied by the solid body, in our case the naturally twisted beam, $n_j = \cos(n_x, O x_j)$, n_x being the unit vector of the outward normal to $\partial \mathcal{B} = S_0 \cup S_h \cup \mathcal{F}$.

In these relations we have used the same notations as in [7]–[12].

3. Statement of the Torsion Problem. Assume that the lateral surface of the beam is free of applied forces and couples and that the load of the beam is distributed over its bases S_0 and S_h in a way which fulfills the equilibrium conditions of a rigid body. Let the loading applied on S_0 be statically equivalent to a force $\mathbf{R} = 0$ and a moment $\mathbf{M} = (0, 0, -M)$.

Therefore, on the base S_0 , we have the following conditions:

$$(6) \quad \int_{S_0} t_{3i} d\sigma_0 = 0, \quad \int_{S_0} (x_\alpha t_{33} - \varepsilon_{3\alpha\beta} m_{\alpha\beta}) d\sigma_0 = 0,$$

$$(7) \quad \int_{S_0} (\varepsilon_{3\alpha\beta} x_\alpha t_{33} + m_{33}) d\sigma_0 = M,$$

and on the lateral surface $\mathcal{F}$, we have

$$(8) \quad t_{ji} n_j = 0, \quad m_{ji} n_j = 0.$$

The torsion problem of the isotropic and homogeneous micropolar elastic naturally twisted beam $\mathcal{B}$ consists in solving equations (2)–(4) with the boundary conditions (6)–(8).

4. Reduction to the Case of Cylindrical Beams. To obtain a more convenient form for the equation of the lateral surface, we introduce new variables defined by [1]

$$(9) \quad \xi_i = x_i + k \varepsilon_{3\alpha i} x_\alpha x_3,$$

which can also be written in the form

$$(10) \quad x_j = \xi_j + k \varepsilon_{3j\alpha} \xi_\alpha \xi_3.$$

Then the lateral surface $\mathcal{F}$ becomes the cylindrical surface $\mathcal{F}_1$ given by $f(\xi_1, \xi_2) = 0$. Therefore, our beam becomes a cylindrical beam $\mathcal{B}_1$ in the space ξ_1, ξ_2, ξ_3 , whose generic cross-section is S_0 , limited by a Ljapunov curve L_0 .

It may be easily proved that

$$(11) \quad \frac{\partial}{\partial x_i} = \frac{\partial}{\partial \xi_i} + k \left(\varepsilon_{ij3} \xi_3 \frac{\partial}{\partial \xi_j} - \delta_{i3} \nabla \right), \quad \text{where}$$

$$(12) \quad \nabla = \varepsilon_{pj3} \xi_j \frac{\partial}{\partial \xi_p} = \xi_2 \frac{\partial}{\partial \xi_1} - \xi_1 \frac{\partial}{\partial \xi_2},$$

and

$$(13) \quad n_\alpha = n_\alpha^* + k \xi_{\alpha p3} n_p^*, \quad n_3 = -k \varepsilon_{pj3} \xi_j n_p^*,$$

where n_α^* are the direction cosines of the outward normal to the curve L_0 .

In this way, equations (2), (4) and (8) become, respectively

$$(14) \quad \begin{cases} t_{ji,j} + k(\varepsilon_{jp3} \xi_3 t_{ji,p} - \nabla t_{3i}) = 0, \\ m_{ji,j} + \varepsilon_{imn} t_{mn} + k(\varepsilon_{jr3} \xi_3 m_{ji,p} - \nabla m_{3i}) = 0, \end{cases} \quad \text{in } B_1;$$

$$(15) \quad \varepsilon_{ij} = u_{j,i} + \varepsilon_{jim} \varphi_m + k(\varepsilon_{ir3} \xi_3 u_{j,p} - \delta_{i3} \nabla u_j), \quad \text{in } B_1 \cup \partial B_1;$$

$$(16) \quad \begin{cases} t_{\alpha i}(n_\alpha^* + k \varepsilon_{\alpha p3} \xi_3 n_p^*) - k t_{3i} \varepsilon_{pj3} \xi_j n_p^* = 0, \\ m_{\alpha i}(n_\alpha^* + k \varepsilon_{\alpha p3} \xi_3 n_p^*) - k m_{3i} \varepsilon_{pj3} \xi_j n_p^* = 0, \end{cases} \quad \text{on } \mathcal{F}_1,$$

where u_i and φ_i are functions of ξ_1, ξ_2, ξ_3 , t_{ij} are related by ε_{ij} , from (15), by the same relations as in (3)₁ and

$$(17) \quad m_{ij} = \alpha \varphi_{r,i} \delta_{ij} + \beta \varphi_{i,j} + \gamma \varphi_{j,i} + k(\alpha \varepsilon_{rpp} \varphi_{r,p} \delta_{ij} + \beta \varepsilon_{jpp} \varphi_{i,p} + \gamma \varepsilon_{ipp} \varphi_{j,p}) \xi_p,$$

in $B_1 \cup \partial B_1$.

From (5)–(7) we see that on the base S_0 of the cylinder B_1 we have the conditions

$$(18) \quad \int_{S_0} t_{3i}(\xi_1, \xi_2, 0) d\sigma = 0, \quad \int_{S_0} [\xi_{\alpha 3} t_{33}(\xi_1, \xi_2, 0) - \xi_{3\alpha 3} m_{33}(\xi_1, \xi_2, 0)] d\sigma = 0,$$

$$(19) \quad \int_{S_0} [\varepsilon_{3\alpha\beta} \xi_\alpha t_{3\beta}(\xi_1, \xi_2, 0) + m_{33}(\xi_1, \xi_2, 0)] d\sigma = M,$$

where $d\sigma = d\xi_1 d\xi_2$, t_{ij} are calculated with ε_{ij} from (15), via (3)₁, and m_{ij} are given in (17).

Therefore, the torsion problem for the naturally twisted beam $\mathcal{B}$ is reduced to the following problem for the cylindrical beam B_1 : to determine $u_i(\xi)$, $\varphi_j(\xi)$, $\xi = (\xi_1, \xi_2, \xi_3)$ so that equations (14)–(19) and (3)₁ be fulfilled.

5. Determination of u_i and φ_j . Guided by our previous works [10]–[12] let us put

$$(20) \quad \begin{cases} u_i(x) = u_i^*(\xi) + k[u_i^0(\xi) + \xi_3 \nabla u_i^*(\xi)], \\ \varphi_i(x) = \varphi_i^*(\xi) + k[\varphi_i^0(\xi) + \xi_3 \nabla \varphi_i^*(\xi)], \end{cases}$$

where u_i^0 , φ_i^0 are unknown displacements and microrotations, respectively, and u_i^* , φ_i^* are the solution of the torsion problem for the cylindrical beam B_1 , given by D. I e ș a n in [8].

Using the same method as in [10]–[12], we obtain for the unknown functions u_i^0 and φ_i^0 the following equations and boundary conditions:

$$(21) \quad l_{ji,j}^0 = 0, \quad m_{ji,i}^0 + \varepsilon_{imn} l_{mn}^0 = 0, \quad \text{in } \mathcal{B}_1;$$

$$(22) \quad \begin{cases} l_{ij}^0 = \lambda \varepsilon_{rr}^0 \delta_{ij} + (\mu + \kappa) \varepsilon_{ij}^0 + \mu \varepsilon_{ji}^0, \\ m_{ij}^0 = \alpha \varphi_{r,r}^0 \delta_{ij} + \beta \varphi_{i,j}^0 + \gamma \varphi_{j,i}^0, \end{cases}$$

$$(23) \quad \varepsilon_{ij}^* = u_{j,i}^0 + \varepsilon_{jim} \varphi_m^0, \quad \text{in } \mathcal{B}_1 \cup \partial \mathcal{B}_1;$$

$$(24) \quad \begin{cases} \int_{S_0} l_{3i}^0 d\sigma = 0, \quad \int_{S_0} (\xi_\alpha l_{33}^0 + \varepsilon_{\beta\alpha\gamma} m_{3\beta}^0) d\sigma = 0, \\ \int_{S_0} (\varepsilon_{\alpha\beta\gamma} \xi_\alpha l_{3\beta}^0 + m_{33}^0) d\sigma = 0, \quad \text{for } \xi_3 = 0; \end{cases}$$

$$(25) \quad \begin{cases} l_{\beta\alpha}^0 n_\beta^* = \varepsilon_{3\gamma\eta} \varepsilon_\eta l_{3\alpha}^* n_\rho^*, \quad l_{\alpha 3}^0 n_\alpha^* = -(\varepsilon_{3\eta\rho} l_{\eta 3}^* + \nabla l_{\rho 3}^*) \xi_3 n_\rho^*, \\ m_{\beta\alpha}^0 n_\beta^* = -(\varepsilon_{3\eta\rho} m_{\alpha\eta}^* + \nabla m_{\rho\alpha}^*) \xi_3 n_\rho^*, \quad m_{\alpha 3}^0 n_\alpha^* = \varepsilon_{3\gamma\eta} \xi_\eta m_{33}^* n_\rho^*, \quad \text{on } \mathcal{F}_1. \end{cases}$$

Equations (21)–(25) define a general problem (see e.g. [6] and [9]) for the cylindrical beam $\mathcal{B}_1$. Taking into account that [8] $l_{3\alpha}^*$, $l_{\beta 3}^*$, $m_{\alpha\beta}^*$, m_{33}^* are functions of ξ_1 , ξ_2 only, we see that this general problem is similarly to those obtained and solved in ([10], 15), by the method presented by C.M. Ieșan in [9].

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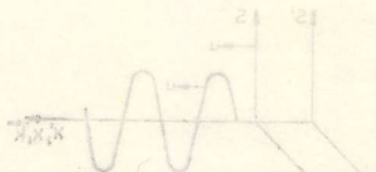
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TORSIUNEA BARELOR MICROPOLARE ELASTICE UȘOR RĂSUCITE, IZOTROPE ȘI OMOGENE

(Rezumat)

Se consideră o bară ușor răsucită alcătuită dintr-un material micropolar elastic izotrop și omogen, pentru care se studiază torsiunea. Această problemă este redusă la rezolvarea unei probleme generale pentru o bară cilindrică, similară celei obținute cu ocazia considerării torsiunii barelor aproape cilindrice.



AN INVARIANT PROPERTY OF THE ALFVÉN PERTURBATIONS

BY

CORNELIU D. CIUBOTARIU and DAVID I. ZOLER

In the nonlinear interaction problems often we are in the situation to choose a system of reference with the following double meaning. Firstly it is necessary to use a referential with an exact physical significance and, secondly, it must lead to a simplified mathematical formalism. In this Note we define the comoving reference system of the Alfvén wave and we find an invariance property of this wave.

This problem is useful, for instance, in the case of the nonlinear decay of an Alfvén wave in two magnetohydrodynamic waves. Let us consider a system of reference S which moves with the speed u together with the Alfvén wave which forms the stationary „background“ of the MHD field (Fig. 1). The vector of the Alfvén wave, k_0 , is parallel with the x'_1 -axis of the system S and with x_1 axis of the reference system S' with respect to which the plasma in general is at rest.

From the invariance of the phase of a plane wave (see, for instance [1]) it results

$$(1) \quad \omega t - k_0 x = \omega' t' - k'_0 x'; \quad x_1 = x, \quad x'_1 = x'$$

and, substituting here the Lorentz transformations, we obtain

$$(2) \quad \omega = 0, \quad k_0 = \gamma \left(k'_0 - \omega' \frac{u}{c^2} \right),$$

where $\gamma = \left(1 - \frac{u^2}{c^2} \right)^{-1/2}$.

Thus, as in the Galileean case too, case described by Orayevsky and Sagdeev [2], the coefficients of the magnetohydrodynamic equations which correspond to the dynamics of the small deviations from the background, in the referential S , do not depend on time. (The magnetic field and the velocity field corresponding to the Alfvén wave have, respectively, the forms $\delta b \cos k_0 x$, $\delta v \cos k_0 x$, and the dependence on time of the solution of these equations can be chosen in the form $\exp(i\omega t)$.)

Let us denote with B_0 , N_0 and P_0 , the homogeneous magnetic field, the density of the number of particles and, respectively, the pressure for the equilibrium state.

D. C. C. 19

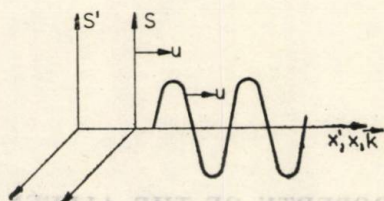


Fig. 1. — The system of reference comoving to Alfvén wave

As for a perturbation to represent an Alfvén wave in the system S' it is necessary that the implied physical variables have the form :

$$(3) \quad \vec{B}'_0 = (B'_{01}, B'_{02}, 0), \quad \delta \vec{B}' = (0, 0, \delta B'_3),$$

$$(4) \quad \delta \vec{V}' = (0, 0, \delta V'_3),$$

$$(5) \quad \vec{E}'_A = (\delta V'_3 B'_{02}, -\delta V'_3 B'_{01}, 0),$$

where we have chosen the uniform magnetic field of the equilibrium state, $\vec{B}'_0$, in the plane $(x'_1, x'_2 = y')$ and $\delta \vec{B}'$ and $\delta \vec{V}'$ are, as it is quite natural, in the case of the Alfvén wave, perpendicular to the direction of propagation $\vec{k}_0$ and to the unperturbed magnetic field $\vec{B}'_0$.

The electromagnetic field tensor which determines the Alfvén wave has, consequently, the form :

$$(6) \quad \overset{A}{H}_{12} = \delta B'_3, \quad \overset{A}{H}_{31} = B'_{02}, \quad \overset{A}{H}_{23} = B'_{01},$$

$$(7) \quad \overset{A}{H}_{41} = \frac{i}{c} \delta V'_3 B'_{03}, \quad \overset{A}{H}_{42} = \frac{i}{c} \delta V'_3 B'_{01}, \quad \overset{A}{H}_{43} = 0.$$

This tensor, with respect to the system S , becomes :

$$(8) \quad \overset{A}{H}_{12} = \gamma \left(\delta B'_3 - \frac{u}{c^2} \delta V'_3 B'_{01} \right), \quad \overset{A}{H}_{31} = \gamma B'_{02},$$

$$(9) \quad \overset{A}{H}_{23} = B'_{01}, \quad \overset{A}{H}_{41} = \frac{i}{c} \delta V'_3 B'_{02},$$

$$(10) \quad \overset{A}{H}_{42} = \frac{i}{c} \gamma (u \delta B'_3 - \delta V'_3 B'_{01}), \quad \overset{A}{H}_{31} = \frac{i}{c} \gamma u B'_{02}.$$

From the equations (8)–(10) and (2) it results that, in the system S , the magnetic field corresponding to the wave-wave interaction may be chosen in the form

$$(11) \quad \vec{B} = \vec{B}_0 + \delta \vec{b} \cos k_0 x + \vec{b},$$

where

$$(12) \quad \vec{B}_0 = (B_{01}, B_{02}, 0),$$

$$(13) \quad \delta \vec{b} = (0, 0, \delta b),$$

and $\vec{b}$ represents the perturbation of the magnetic field corresponding to the new waves coupled with the Alfvén wave of the stationary background.

For the Alfvén perturbations of the velocity in the system S , from (4) we obtain

$$(14) \quad \delta \vec{V} = \left(-u, 0, \frac{1}{\gamma} V'_3 \right)$$

and the velocity of the plasma element in the case of the wave-wave interaction is

$$(15) \quad \vec{V} = \vec{V}_0 + \delta \vec{v} \cos k_0 x + \vec{v},$$

where

$$(16) \quad \vec{V}_0 = (-u, 0, 0), \quad \delta \vec{v} = (0, 0, \delta v).$$

So, from the equations (3)–(5) and (12)–(16) it results the following theorem: *The Alfvén wave defined in an inertial system S' keeps its character of an Alfvén wave in the comoving system S too.*

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O PROPRIETATE DE INVARIANTĂ A PERTURBAȚIILOR ALFVÉN

(Rezumat)

Se arată, pe baza transformării relativiste a fazei unei unde plane, că o undă Alfvén definită într-un sistem inerțial își păstrează caracterul de undă Alfvén și în sistemul comobil cu ea.

METHOD AND EQUIPMENT FOR MEASURING DYNAMIC VISCOSITY OF FERROFLUIDS

BY

E. LUCA, I. STOICOIU, V. IUSAN and V. MUNTEAN

1. Introduction. Viscosity represents a very useful research method for dispersion studies owing to the information contained regarding the nature of the dispersed phase [1], to the interactions between the particles of the dispersed phase and the medium of dispersion [1], or to the interactions between the particles dispersed outside or within a magnetic field [2].

The literature referring to this subject points out some methods for measuring viscosity. In this paper we shall present the results obtained in determining dynamic viscosity of magnetite dispersions in organic liquids (ferrofluid), making use of the Höppler method improved by an original proceeding designed to detect the position of the ball with the help of ultrasounds.

2. The Device of the Experiment. The equipment for measuring the viscosity of the dispersions under consideration comprises a Höppler viscosimeter to which a thermostat and a DI-4T ultrasonic defectoscope has been attached.

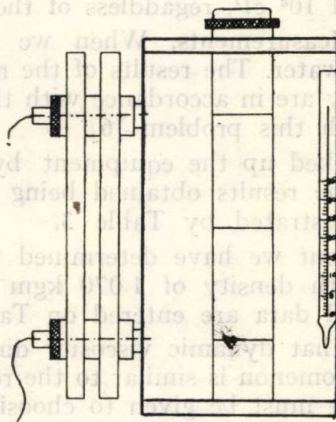


Fig. 1

Since solid dispersions in liquids are generally nontransparent detection of the ball is achieved by means of two palpators connected in pa-

rallel. This detection is based upon the reflection of the ultrasonic beam. The palpators are set on a metallic frame at a distance of 100 mm from one another. They are pressed upon the exterior wall of the Höppler device by means of a mechanic device with a spring. In order to facilitate transmission of the ultrasonic signal we have laid a thin film of vaseline between the wall of the device and the palpators.

The ultrasonic signal is transmitted by the palpators (Fig. 1), it penetrates the glass wall, the heating bath, the glass wall, the liquid of unknown viscosity, the glass wall again etc., which undergoes the reflection phenomenon, the reflected signals being visualized on the defectoscope screen.

The passing of the ball causes a diminution of the signal for the second time reflected on the glass wall, a fact which can be noticed on the screen of the defectoscope.

The position of the ball in front of the palpator can be established not only optically (by means of the observations made on the screen of the defectoscope) but also acoustically with the help of the acoustic signal produced by the defectoscope, as the passing of the ball in front of the palpator discontinues the audio signal. In order to determine the span of time taken by the ball to fall between the two palpators we use a chronometer which is started and stopped by the optical or acoustic signal. Dynamic viscosity is computed with the help of the following relation [5]:

$$\eta = k(\rho - \rho_e t),$$

where: η — the dynamic viscosity in cP; ρ — the relative density of the ball; ρ_e — the relative density of the fluid of unknown viscosity; k — the constant of the ball; t — the falling time in seconds.

Using this equipment one can measure viscosities of fluids within the range of 0.5 cP and 10⁶ cP, regardless of their transparency.

3. Experimental Measurements. When we have checked up the device we used distilled water. The results of the measurements have been entered on Table 1; they are in accordance with those put forward by the literature concerned with this problem [6].

We have also checked up the equipment by determining viscosity of certain substances, the results obtained being close to those known [6], a fact which is illustrated by Table 3.

Using this equipment we have determined viscosity of a ferrofluid based on petroleum, with density of 1070 kgm⁻³ depending on temperature. The experimental data are entered on Table 2.

We have noticed that dynamic viscosity diminishes when the temperature rises. This phenomenon is similar to the results obtained by other authors [7]. Utmost care must be given to choosing the ball which must correspond to the domain of laminar running, otherwise a turbulent current may appear which may modify the experimental result. We assert that the ball is proper if the falling interval is more than 25 s. and less than 300 s. for the distance of 100 mm between the palpators.

Table 1

Experimental data for distilled water obtained by means of palpators device

Nr.	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Temperature [C°]	10	15	20	25	30	35	40	45	50	55	60	65	70	75
Time [s]	91,3	79,9	70,4	62,5	55,2	50,1	45,7	41,5	38,3	35,2	32,4	30,7	28,9	26,2
η [cP]	1,298	1,136	1,006	0,888	0,798	0,712	0,650	0,592	0,544	0,500	0,460	0,436	0,398	0,372
<i>The value of η for distilled water in the field literature</i>														
η [cP]	1,307	1,138	1,002	0,898	0,797	0,719	0,653	0,598	0,548	0,505	0,467	0,434	0,404	0,378

Table 2
Experimental data for a ferrofluid with $\rho = 1\,170\text{ kgm}^{-3}$

Nr.	1	2	3	4	5	6	7	8	9	10	11	12
Temperature [C°]	18	20	22	25	27	30	32	35	37	40	42	45
Time [s]	44,8	41,1	39,1	37	35,4	34,2	33,1	31,4	29,8	28,1	27,2	25,4
η [cP]	4,085	3,747	3,574	3,373	3,227	3,010	3,018	2,863	2,717	2,553	2,480	2,316



Table 3

Dynamic viscosity for various substances obtained with palpators device

Substance	Temperature [C°]	Time [s]	η [cP]	η in literature
Ethyl alcohol	18	86,5	1,231	1,220
Aniline	18	44,8	4,635	4,620
Toluene	18	42,9	0,620	0,613
Benzene	18	47,1	0,677	0,673

4. **Conclusions.** 1. This equipment can measure dynamic viscosity of turbid or even totally nontransparent solutions for which detection of the ball is difficult or impossible to be achieved with one's look.

2. The ultrasonic signals can detect the ball with utmost accuracy avoiding thus the drawback mentioned above. In this way we can measure dynamic viscosity of any fluid regardless of its transparency.

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METODĂ ȘI INSTALAȚIE PENTRU MĂSURA REA VÎSCOZITĂȚII DINAMICE A FEROFUIDELOR

(Rezumat)

Se prezintă o metodă și o instalație de determinare a viscozității lichidelor ne transparente utilizând traductori ultrasonori. Se compară rezultatele obținute pe lichide cunoscute cu datele din literatură.

CdS/Cu₂S THIN FILM CELLS

BY

SOFIA MELINTE and ALEXANDRIŢA JEFLEA

1. Introduction. Thin film solar cells have become more and more competitive in the past few years over the silicon cells. Of the former, the polycrystalline heterojunction CdS/Cu₂S cells are being investigated more extensively and are exhibiting promising results. As compared to those made from monocrystalline semiconductor materials, thin film solar cells have certain advantages, namely: their manufacturing technology is relatively simple; their cost is low since they require small quantities of both the base and the noble materials for their manufacture; since both the interface width and the base layer thickness are small, the total photoactive surface obtainable can be developed over a large area.

As can be seen, these advantages are more important than their disadvantages, such as their rather low efficiency, and their instability.

This paper gives a description of manufacturing aspects of several thin film CdS/Cu₂S heterojunction solar cells, as well as a review of their characteristics. It also provides a discussion of the parameters controlling the short-circuit current, the open-circuit voltage, and the fill factor, and appraises the limits obtained for each of these factors. The effect thermal treatment has on spectral response is also discussed, as are also the current-voltage characteristics. The paper also identifies those processing parameters which must be improved in order to improve cell efficiency.

The aim of our research was to obtain increased efficiency and better stability by means of the technology used and of the applied treatment.

The cell manufacture technology is the one described in [1], [2], [3]. The solar cell is made up of a metallic underlayer of electrolytic copper, a zinc layer over which CdS in powder form is deposited by evaporation under vacuum, and then Cu₂S by chemical reaction, using a thermodynamically stable CuCl-NH₄Cl solution. A 50 Å thick copper layer is then superimposed, followed by the grid and the gold contacts.

A thermal treatment was then applied for about 16 hours under vacuum and atmospheric conditions, the resulting cell stability being good.

2. The Factors Controlling the Cell Parameters (I_{sc} , V_{oc} , FF). For a given solar cell, its maximum power is given by the relation

$$P_{\max} = I_{sc} V_{oc} FF = j_{sc} A_{\perp} V_{oc} FF,$$

where j_{sc} is the short-circuit current density, $A_{\perp}$ is the cell area, V_{oc} is the open-circuit voltage, which depends on the applied thermal treatment and illumination, and FF is the fill factor. While being interdependent, these three factors also depend individually on other specific factors [2], [4].

2.1. The Short-Circuit Current. The equation giving the density for the current flowing through a given cell [2], [4] is

$$(1) \quad j_{sc} = \frac{\mu_2 E_2(\Phi')}{v_r + \mu_2 E_2(\Phi')} \int_{\lambda=0}^{\lambda_g} \Phi_0(\lambda) T_g [1 - R(\lambda)] [1 - A(\lambda)] \eta_{col} [\alpha(\lambda), L, d_1, E_1, r, v] d\lambda.$$

It follows that the current density depends on: μ_2 — the mobility of the charge carriers in CdS; the electric field of the junction E_2 , which depends on the total photon flux reaching the CdS/Cu₂S interface; $\lambda_g = \hbar c / E_g$ — the wavelength associated with the forbidden zone for Cu₂S; the density of the photon flux reaching the cell $\Phi_0(\lambda)$; the transmission of the grid contact T_g ; the reflexion coefficient $R(\lambda)$, and the absorption coefficient $A(\lambda)$; the collecting efficiency η_{col} of the Cu₂S layer.

It can be seen that collecting efficiency depends on the absorption coefficient $\alpha(\lambda)$, the diffusion length L , the Cu₂S layer thickness d_1 , the drift field E_1 , the grain size, and the superficial recombination velocity.

While the process of layer formation is under way, it is difficult to control such a variety of factors and to obtain the desired current density. We cannot know certain factors, such as $\alpha(\lambda) \rightarrow$ and L , which depend on the stoichiometry of the Cu₂S layer — in fact Cu_xS —, and which lead to good results only for $x > 1.99$.

It is possible to avoid certain losses, as such those caused by superficial recombination, by forming the Cu₂O layer on the Cu₂S layer, but the volume losses are essentially about 15% [6].

Another loss-causing factor results from photon absorption by the CdS layer, caused especially by those photons exhibiting an energy $h\nu < E_{g2}$ (the CdS forbidden zone energy).

2.2. The Open-Circuit Voltage. The open-circuit voltage V_{oc} can be obtained from

$$(2) \quad qV_{oc} = E_{g1} - \Delta x + kT \ln j_{sc} - kT \ln q N_{c2} v_r - kT \ln (A_j / A_{\perp}),$$

where q — the electron charge; A_j — the junction area, considerably larger than its cross-section due to the formation of the Cu₂S layer; $N_{c2} \cong (2 \cdot 10^{18} \text{ cm}^{-3})$, as reported in [2], represents the effective state density in CdS; $v_r \approx 10^6 \text{ cm/s}$ [5]; $E_{g1} \cong 1.2 \text{ eV}$ for Cu₂S; Δx — the difference in affinity between Cu₂S and CdS, and equals about $\approx 0.2 \text{ eV}$.

An important improvement in the open-circuit voltage can be obtained by modifying Δx , v_r and A_j . Since Δx and v_r depend on the material used, by selecting other materials, we can modify these factors. Also, we can modify A_j , the junction area, by selecting a plane section, instead of a spiralling one. This can lead to a gain of 0.06 V for V_{oc} [2]. The junction can be built by dry evaporation of cuprous chloride (Cl₂Cu₂).

2.3. *The Fill Factor FF.* The form factor or fill factor constitutes another quantity which influences the cell efficiency. It, too, depends on other factors, as can be obtained from the following relation

$$(3) \quad FF = - \frac{c}{V_{oc}} j_{sc} R_s A_{\perp}$$

where $c=0.9$, R_s is the series resistance of the cell which varies with the Cu_2S layer's resistivity and thickness, as seen from the following relation [6]

$$(4) \quad R_s = \frac{\rho_1 s^2}{12 d_1 A_{\perp}},$$

where ρ_1 is the Cu_2S layer resistivity, d_1 — the thickness of this layer, and s is the grid spacing.

From what we have presented up to now it can be seen that the parameters controlling the solar cell's efficiency depend on a great variety of factors, which have been analyzed extensively in [2], [3], [5], and which on one hand cannot be controlled in the technological process, and on the other lead to considerable current and voltage losses. Certain parameters can be improved by means of thermal treatments.

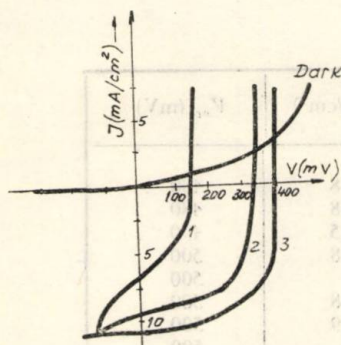


Fig. 1 — The $J-V$ characteristic of the same cell after successive heat treatments at 170°C , in the air ($h = 0.5$ — curve 1; $h = 1.5$ — curve 2 and $h = 2.5$ — curve 3).

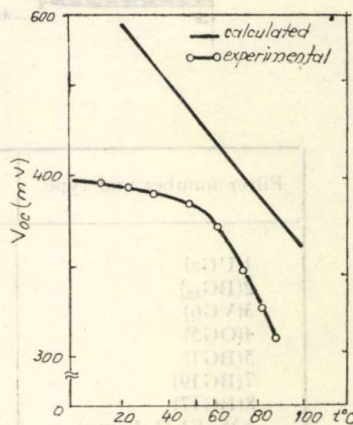


Fig. 2

Fig. 1 shows the considerable influence thermal treatments have over the efficiency-controlling parameters of the solar cell. It is to be noted that both the current density, and the open-circuit voltage increase with the treatment duration. The thermal treatment was first conducted in vacuum, and then in the atmosphere.

Fig. 2 gives the open-circuit voltage variation with the cell's working temperature. Up to about $\approx 50^\circ\text{C}$ the open-circuit voltage V_{oc} decreases very small with the increase in temperature, showing this temperature range to have no effect on the efficiency of the cell.

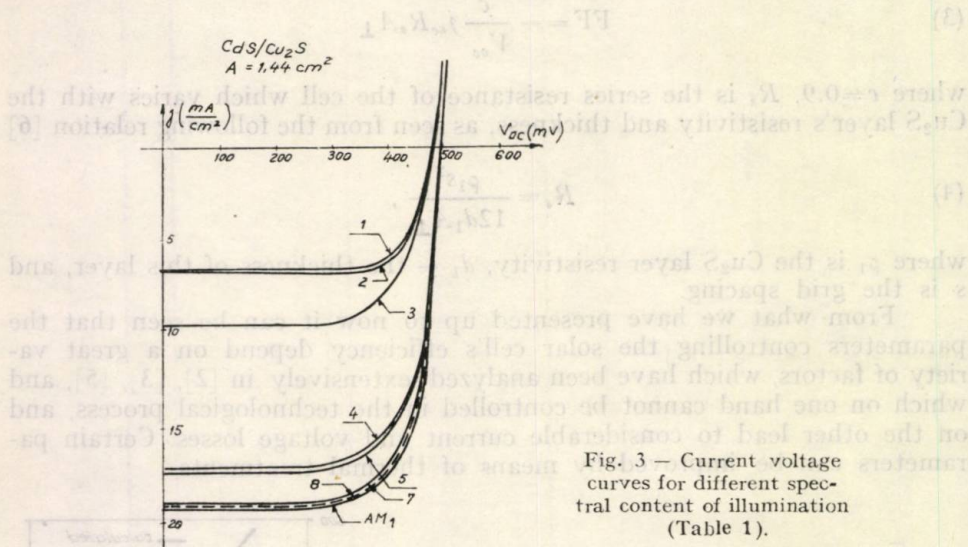


Fig. 3 — Current voltage curves for different spectral content of illumination (Table 1).

Table 1

Filter number and type	$\lambda(\text{nm})$	$j_{sc} (\text{mA}/\text{cm}^2)$	$V_{oc} (\text{mV})$
1(UGz)	320—400	6.8	480
2(BG ₁₂)	400—480	6.8	480
3(VG6)	480—570	9.5	490
4(OG5)	560—600	16.8	500
5(BG1)	600—650	17	500
7(BG19)	650—820	18.8	500
8(BG17)	660—890	18.9	500
AM ₁ (ELH Lamp)	—	19	500

Fig. 3 gives the influence of the spectral contents over the CdS/Cu₂S cell efficiency. The current density exhibits a maximum in the wavelength range $\Delta\lambda=570-650$ nm, and the open-circuit voltage has its maximum for $\Delta\lambda=650-890$ nm.

The resulting spectral range offering maximal efficiency can thus be seen to be sufficiently large (Table 1).

4. Conclusions. 1. The efficiency of the CdS/Cu₂S solar cell is influenced by a very large number of factors that are difficult to control during the manufacturing process.

2. The thermal treatment applied improves the cell stability, as well as the parameters controlling the performance of the cell. If under working conditions the cell becomes heated, a decrease in its voltage output will result.

3. The spectral contents of radiation will have an influence over the performance of the cell.

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CELULE SOLARE TIP CdS/Cu₂S CU STRATURI SUBȚIRI

(Rezumat)

Se prezintă fabricarea și caracterizarea celulelor solare de tip CdS/Cu₂S în strat subțire. Se determină parametrii ce controlează curentul de scurtcircuit, tensiunea în circuit deschis și factorul de umplere. De asemenea se tratează efectul tratamentului termic asupra răspunsului spectral și asupra caracteristicii curent-tensiune.

Bi₂Te₃ THIN FILM THERMOELECTRIC GENERATORS

BY

ALEXANDRINA JEFLEA and SOFIA MELINTE

1. Introduction. For a more efficient utilisation of the solar energy, thermoelectric converters were made of thin semi-conducting films that are able to work both independently and in association with photo-voltaic cells, recovering the heat and cooling them at the same time.

The thermoelectric converters are already used on a large scale for the conversion of the thermal energy into electric energy [1], [2]. While their working principle, based on the Seebeck effect, was already known for a long time, the utilization of these converters became possible only after producing and spreading the semi-conducting materials that have a Seebeck factor higher than that of the metals. The materials used in constructing the generators are chosen considering the temperature range where the generator is to work.

In the average temperature range from 50 to 150°C, which can be obtained in a concentration regime with a small concentration factor (with plane collectors), the materials having the highest thermoelectric figure of merit Z are those based on Bi₂Te₃ [3], [4].

2. Thin Film Preparation from the Bi₂Te₃-based Thermoelectric Materials. Bi₂Te₃ gives isomorphous solid solutions with both Se₂Bi₃ and Sb₂Te₃, meeting the conditions concerning the crystal lattice and atomic dimensions; in other words:

i) they belong to the symmetry group D_{3d}^5 ($R\ 3m$) with the lattice parameters ranging between 4.134 Å and 4.3835 Å for a , and 28.546 Å and 30.490 Å for c [5];

ii) the radii of the covalent atoms change between 1.17 Å for Se and 1.51 Å for Bi, so that the atoms can be replaced by one another without modifying the crystal lattice parameters [3].

The mixtures of 75% Sb₂Te₃ and 25% Bi₂Te₃ doped with Pb and 80% Bi₂Te₃ plus 20% Bi₂Se₃ doped with CdCl satisfy the imposed conditions and we have chosen them as basic materials in producing the converters.

The problem of obtaining thin films made of these composed materials is especially difficult because, as most of the complex materials, during the evaporation process they dissociate themselves into the constituent elements, which have quite different evaporation and deposition

rates (ranging between 10^9 – 10^{18} mol/cm²sec at a temperature of 520°C and a pressure under 10^{-2} Torr) 6 and the film obtained by vacuum thermal evaporation does not obey the stoichiometry of the material.

In order to avoid these difficulties, flash and cathodic evaporation methods were suggested [5] for producing thin films, since these materials are able to transfer the material from the evaporation source to the substrate without changing the composition.

3. Preparation and Performances of Thin Semi-conducting Films.

The thin films were obtained by means of the device schematically presented in Fig. 1, which permits that in the superheated evaporator only a small quantity of material should reach, which evaporates instantaneously, without dissociation. In order not to affect the purity of *p*- and *n*- type films, two separate devices were used. The bulk materials were delivered by I.C.P.I.A.F. Cluj-Napoca, strictly observing the composition. During the evaporation, a vacuum of $2 \cdot 10^{-5}$ Torr was maintained.

Polyethylene terephthalate leaf and glass slips for the microscope were used as substrates. The substrate was not additionally heated during the deposition.

The preparation of thin films with reproducible properties was performed by following their resistance during the deposition, i.e. controlling their thickness. The thickness of the prepared films was measured by means

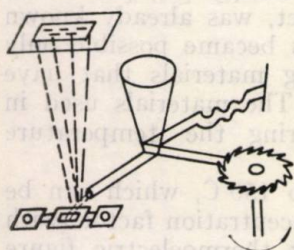


Fig. 1

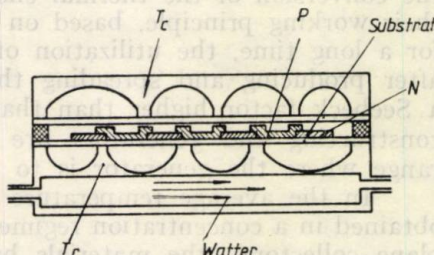


Fig. 2

of an interference microscope. We prepared films with thicknesses up to 15 μ m.

After preparation, the films deposited on glass were submitted to a thermal treatment which improved the thermoelectric properties of the films by the recrystallization of the layer and the sublimation of the constituents of the fifth group in excess.

From the measurements that we have made on a couple of *p*-*n* thin films, we have found a thermoelectric force of 0.2708 mV/°C, in comparison with 0.34–0.36 mV/°C for the bulk material. For the films that were thermally treated at 175°C for two hours, the electric resistance decreased from 45 Ohms to 27 Ohms, i.e. their resistivity changed from $36 \cdot 10^{-3}$ Ohms. cm to $1.92 \cdot 10^{-2}$ Ohms. cm, as opposed to $(1.1-0.91) \cdot 10^{-3}$ Ohms. cm for the bulk material.

The acquirement of films with these performances permitted the realisation of thermoelectric generators.

4. Accomplishment of Thin Thermoelectric Generators. In the case of thin film thermoelectric generators, we have the possibility of arranging the thermoelectric couples in different ways, depending on the conditions imposed by the optimum operation of the generator [6]. Taking into consideration that the plane thermoelectric generator is characterized by an important heat transfer, this is preferably used for cooling the photovoltaic cells and heating the water.

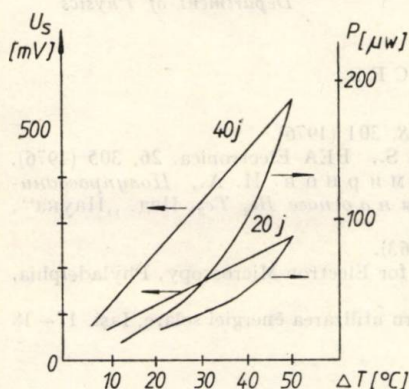


Fig. 3

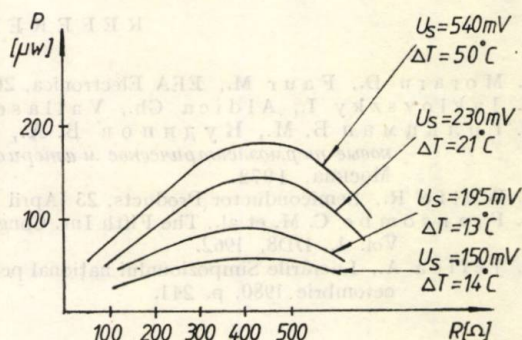


Fig. 4

The heat is taken over by means of a copper lugged piece situated above the warm junctions T_c of the thin film. In dimensioning the generator, we have taken into account the thin film thickness and the interval between the junctions (under the conditions of an isotropic thermal conductivity of the film). A similar piece is cooling the cold junction T_r of the thin film by means of water circulation.

The heat transfer in this case is done in the direction perpendicular to the surface of the film, the transferred heat being limited only by the conductivity of the substrate. It is preferable that the transferred heat should be taken over by a cooling agent (see Fig. 2).

In the following, we shall describe the performances of such a generator. Our thermoelectric generator is doing the thermal difference of 50°C , at $T_{\text{warm}} = 66^\circ\text{C}$. Under these circumstances the thermoelectric voltage was of 540 mV for 40 junctions and 270 mV for 20 junctions, with an electric power of 192 μW and 96 μW respectively (see Fig. 3).

The Fig. 4 gives the load characteristic for the thermoelectric voltages of 540 mV, 230 mV, 195 mV and 150 mV, corresponding to the output powers of 192 μW , 140 μW , 98 μW and 60 μW respectively, for a load resistance of 400 Ohms.

5. Conclusions. Even if the thermoelectric voltage and the power of such a generator are still small, the perspectives opened for the thin film thermoelectric generators are optimistic, they exhibiting a series of important advantages, as follows:

- i) small consumption of thermoelectric material;
- ii) the possibility of miniaturization and hence a great number of junctions can be series or parallelly connected on a narrow space;
- iii) they can be used for the conversion of the solar energy by themselves or associated with other types of converters, taking over from them only the heat;
- iv) they can also be used in other fields, recovering the waste heat.

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GENERATOR TERMOELECTRIC PLAN CU STRATURI SUBȚIRI PE BAZĂ DE Bi_2Te_3

(Rezumat)

Se prezintă tehnica de obținere a straturilor subțiri termoelectrice pe bază de Bi_2Te_3 și soluțiile sale solide cu Bi_2Se_3 și Sb_2Te_3 prin metoda evaporării instantanee, proprietățile straturilor și caracteristicile de funcționare ale unui generator realizat.

Ca posibilități de utilizare ale generatorului termoelectric plan în conversia energiei solare se subliniază utilizarea lui în conversia directă în regim de concentrare mic și cuplat cu o celulă fotovoltaică ca recuperator de căldură deșeu.